When does aggregation reduce risk aversion?[*]

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Abstract

We study the problem of risk sharing within a household or syndicate. A household shares risky prospects using a social welfare functional. We characterize the social welfare functionals such that the household is collectively less risk averse than each member, and satisfies the Pareto principle and an invariance axiom. We single out the sum of certainty equivalents as the unique member of this family which is quasiconcave over riskless allocations.

1   Introduction

This paper is devoted to the sum of individual certainty equivalents, a common method of aggregating individual preferences into collective welfare in

an environment of idiosyncratic risk. Each agent in a household[1] faces risk over her individual monetary consumption. The sum of individual certainty equivalents for a given allocation of risky prospects is the sure amount that the household would need to be paid in order to give up the allocation (without first reallocating).

We show that the sum of individual certainty equivalents possesses a property singling it out from other methods of aggregating welfare. That is, with this rule, a household of risk averse individuals will behave in the aggregate in a fashion that is *less risk averse* than each member of the household. The notion that a group is, or should be, less risk averse than its members is a very familiar one in economics. Early arguments for the notion are in Samuelson (1964), Vickrey (1964) and Arrow and Lind (1970).

The environment is simple. Each member of a household faces risky consumption of a single good, say money. Members of the household have differing attitudes toward risk. Given an aggregate risk, the household seeks to allocate the risk to its members to maximize some notion of social welfare.

Our main result can be roughly stated as follows: Suppose that the household ranks allocations using a social welfare functional $\succeq^0(\mathbf{R})$, which depends on individual preferences $\mathbf{R} = (R_i)_{i \in N}$. The *sum of individual certainty equivalents* represents the only social welfare functional (SWF) that

1. generates households less risk averse than its members (for all individual preference profiles);

2. ignores risk preferences whenever it compares riskless allocations;

3. is quasiconcave over riskless allocations.

This result is a simple consequence of a theorem stating that (1) and (2) are equivalent to ranking allocations with a function which takes as input

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[1]We use "household" throughout to refer generically to a group of agents engaged in a risk sharing arrangement. Wilson (1968) uses the term "syndicate."

certainty equivalents, and when maximized over a simplex, has a solution at every corner. In particular, in any such social welfare functional, a riskless amount should always all be given to one member of the household–but which member does not matter. This theorem, in turn, uses some classical results on aggregation from demand theory (concretely Samuelson (1956) and Chipman and Moore (1979)). Finally, (3) singles out the sum of individual certainty equivalents among this class.

The sum of certainty equivalents has a simple economic interpretation: it is the certain monetary sum that the household would demand for an allocation of risky prospects. In other words, it is the collective willingness to accept for an allocation. Myerson (2004), for example, recommends MBAs and applied decision makers use the sum of certainty equivalents to share risk. The recommendation is based on the idea of maximizing the collective willingness to accept for an allocation. Our paper provides a completely different justification for this recommendation. A normative justification of the sum of certainty equivalents would is based on the desirability of axioms (1)- (3). We expand on these axioms below.

Other rules are commonly used in welfare analysis; arguably on the grounds that they are more tractable. We show that, in fact, the sum of individual certainty equivalents yields very tractable results. In the specialized setting of convex homothetic preference profiles (such as CRRA preferences), it yields the most risk averse *convex* household preference, among those which are less risk averse than members' preferences. This result lends itself to a simple representation using basic convex duality.

We proceed to discuss our results in more detail.

We suppose a finite set of states of the world. A prior distribution over these states is exogenously specified. Agents' preferences are over state-contingent monetary payoffs, which we call acts. We impose little structure on the preferences of agents other than monotonicity and a weak notion of

risk aversion; in particular, agents need not be expected utility maximizers.

A SWF recommends a preference over allocations for any given list of individual preferences. We impose the Pareto principle: the SWF must be increasing in the welfare of individuals.

We present two new axioms. The first axiom is an invariance axiom: individuals' risk preferences should not matter for comparing two riskless allocations; an allocation is riskless when each agent is allocated a constant act–an act whose payoff does not depend on the state. In comparing two riskless allocation, one must trade off more money for some individuals and less for others. Note that individual preferences do not differ over riskless allocations, they are monotonic. The axiom states that the tradeoffs should be resolved without regard for the agents' risk preferences. This axiom requires that we do not interpret the degree of risk aversion as being related to a "marginal utility of income." This is merely a philosophical hypothesis on our part: several authors do suggest that the degree of risk aversion can be interpreted as a marginal utility of income, and in fact there is neither a way of proving nor disproving this in our model.[2] In particular, one should consult **?** and **?**.

Our second and main new axiom is reduction of risk aversion. It says that the household is less risk averse than its members. The axiom requires defining a comparative notion of risk aversion. Following Yaari (1969), we say that a preference R_1 is less risk averse than R_2 if, for every constant act c and every act x, when $x\,R_2\,c$ then $x\,R_1\,c$. The idea is that if the risk involved in choosing the risky act x over the certain act c is acceptable for R_2, then it must also be acceptable for R_1.

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[2]We believe the relevant data for understanding marginal utility of income are data on changes in income. That is, comparisons across objects of the form (x, y) would need to be considered, where (x, y) represents a change in income, from x to y. These types of choices are outside of our model.

The social welfare functional guides the household's decisions on how to share risky prospects. In a classic paper discussing the representative consumer problem in demand theory, Samuelson (1956) (see also Graaff (1957), p. 49) shows that a household which allocates aggregate bundles optimally according to some SWF behaves as if it is an individual (that is, it has a complete and transitive preference). Our social welfare functional generates such a preference for each list of individual preferences–this is what we call the *household preference.* Our main axiom requires that this household preference be less risk averse than the preferences of each member of the household.[3]

The assumption that a government or a firm behaves in a risk neutral fashion is often justified on the grounds that large groups of agents will tend to behave in a risk neutral fashion. It is understood that the second order effects of risk can be mitigated by properly sharing risk. However, it is equally clear that not *all* methods of risk sharing will have this effect; for a simple counterexample consider a dictatorship. Our requirement of reduction of risk aversion is a simple fixed-population notion capturing the intuition that societies should tend toward risk neutrality.

Normative arguments for reduction in risk aversion are familiar in economics (Samuelson, 1964; Vickrey, 1964; Arrow and Lind, 1970). The idea has been that less cautious groups may gain from larger expected payoffs, and internally share the additional risk. There is also a strategic reason for reduction in risk aversion: a less risk averse agent may fare better in bargaining (Rubinstein, Safra, and Thomson, 1992).

Alternatively, we know that in environments of inter-household bargain-

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[3]Household preferences result from individual preferences aggregated by means of the social welfare functional. The household will optimally decentralize an aggregate prospect in accordance with the SWF: as a result of this optimization one obtains a ranking over aggregate bundle, this ranking is the household's preference.

ing, the less risk averse a household, the more rewards a household will obtain. Reduction of risk aversion then states that it is better for the household to act collectively than to be represented by any single individual. Hence, it is reminiscent of an "incentive compatibility" condition on SWF's.

We characterize the family of SWF's satisfying these axioms. The sum of individual certainty equivalents represents the unique one which is *quasiconcave over riskless allocations*. That is, for each individual's state-contingent consumption, the rule finds the certain amount that the individual would need to be given in compensation, then adds these across individuals.

More generally, we refer to the only rules which satisfy our axioms as *anyone can take all* (ACTA) rules. These rules are characterized by a kind of social utility function. The social utility has the property that for any allocation, social utility is a function only of the certainty equivalents of that allocation. Further, the rule is called an ACTA rule as for any riskless amount, it is deemed socially optimal to give the entire amount to any one of the individuals in the household.

1.1   Related literature

There is a vast literature on risk-sharing in economics. Seminal papers discussing optimal risk sharing include Borch (1962) and Wilson (1968) (see also Chateauneuf, Dana, and Tallon (2000)). Under the assumption that all agents are subjective expected utility maximizers, they determine that, under certain conditions (risk aversion or a continuum of states) all Pareto optimal allocations can be obtained by maximizing a weighted sum of subjective expected utilities.[4] A central result of Wilson (1968) is that the risk tolerance

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[4]In particular, under these assumptions, Pareto optimal allocations satisfy what Gollier (2001) terms the "mutuality" principle–consumption of each individual depends only on the aggregate amount in each state. As the sum of certainty equivalents satisfies the Pareto principle, any allocation it recommends is Pareto optimal and hence satisfies the

of household preference is the sum of risk tolerances of each individual *at the optimal household consumption.*[5]

We present our results in a framework with general "non-expected utility" preferences. We do so because the results are more transparent this way, but our theorem is general enough to apply to most decision theoretic models existing in the literature, including (but not limited to) Schmeidler (1989), Gilboa and Schmeidler (1989), Machina and Schmeidler (1992), Klibanoff, Marinacci, and Mukerji (2005), Maccheroni, Marinacci, and Rustichini (2006), Siniscalchi (2007), Ergin and Gul (2008), Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2008), and Seo (2008). In the paper, we have used the expression "risk aversion," although probabilities here could obviously be taken to be subjective (and common). In this sense, we operate in a general framework of Knightian uncertainty. The details are spelled out in ?, the working paper version of the present paper.

Our work differs from previous studies concerning risk in that it is normative. Most previous studies seek to explain behavioral phenomena in markets, for example see Dow and Werlang (1992), Epstein and Wang (1994), Epstein (2001), Rigotti and Shannon (2005) and Rigotti, Shannon, and Strzalecki (2008). In contrast, we try to understand the SWF that satisfy normatively appealing axioms. In that sense, the exercise is closer to Wilson (1968).

Section 2 provides the model; Section 3 has the main results; Section 4 presents results for homothetic preferences, and examples of familiar special cases. Section 5 provides discussion and related literature. Proofs are

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mutuality principle.

[5]Gollier (2001) builds on this result, showing that if all individuals have identical preferences, then a weighted utilitarian planner who optimizes social welfare given a constraint on *average* consumption results in a less risk-averse household preference if and only if the individual risk tolerance is convex. It should be noted that this is a fixed-profile result: the weighted utilitarian rules applied to arbitrary subjective expected utility profiles do not typically reduce risk aversion.

collected in an appendix.

2 The model and definitions

Let Ω be a finite set of **states** of the world. The distribution of the states is governed by an exogenous probability measure π which has full support. **Acts** are state-contingent elements of $\mathbb{R}_+$; that is, the set of acts is $X = \mathbb{R}_+^\Omega$. Let $N = \{1, ..., n\}$ be a finite set of agents. An **allocation** is an element of X^N. An allocation of $x \in X$ is a vector $\mathbf{x} = (x_1, ..., x_n) \in X^N$ for which $\sum_{i \in N} x_i = x$. A **constant act** $c \in X$ is an act which takes only one value. A **constant allocation** is an allocation of constant acts.

A **risk averse preference relation** R is a complete, transitive, continuous, and monotonic[6] binary relation on X satisfying $(\pi \cdot x) \, R \, x$ for all $x \in X$.[7] The latter requirement is that R is risk averse.[8] The set of preferences is denoted $\mathcal{R}$. A **preference profile** is a vector $\mathbf{R} = (R_1, ..., R_n) \in \mathcal{R}^N$. N-vectors are written in boldface throughout.

> **Remark:** *While our story involves risk with known probabilities π, our notion of a risk averse preference relation can accommodate a host of generalized expected utility theories. Suppose for example that there exists a strictly increasing and concave $u : \mathbb{R}_+ \to \mathbb{R}_+$ and increasing continuous $W : \mathbb{R}_+^\Omega \to \mathbb{R}$ for which for all $x \in \mathbb{R}_+$, $W(y) \geq W(x, \ldots, x)$ implies $\pi \cdot y \geq x$. Then, the preference R on x, y given by $x \, R \, y$ if and only if $W(u(x(\omega))) \geq W(u(y(\omega)))$ satisfies the condition required to be a risk*

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[6]That is, if $x(\omega) > y(\omega)$ for all $\omega \in \Omega$, then x is strictly better than y for R ($x \, R \, y$ and not $y \, R \, x$).

[7]We here abuse notation in a standard way by identifying the constant act which takes value c in every state with c itself.

[8]This is a relatively weak notion; one might also ask that the preference is averse to mean-preserving spreads; but this adds nothing to our analysis.

averse preference relation. This condition requires simply that π support the upper contour set of W at any point having equal coordinates.

To see that this condition is sufficient, fix some $x \in X$ and suppose by means of contradiction that $\pi \cdot x \, R \, x$ is false. Then in particular, we know that $W(u(x(\omega))) > W(u(\pi \cdot x))$. There is some ω for which $u(x(\omega)) > 0$ (otherwise we must have indifference), so by continuity and monotonicity of W we can actually establish that $\pi \cdot (u \circ x) > u(\pi \cdot x)$ (using the fact that π is full support and satisfies the condition described above). This is a contradiction to concavity of u.

A typical example would involve a "multiple priors" decision maker, where u is a concave function and $W(x) = \min_{p \in \Pi} p \cdot u \circ x$, where $\pi \in \Pi$ is required for the condition to be satisfied. To see this, suppose that $\pi \in \Pi$ and that $W(x) \geq W(y, \ldots, y)$. Then we must have $\pi \cdot x \geq \min_{p \in \Pi} p \cdot x \geq y$.

Thus, our notion of a "prior" is extremely weak in a decision theoretic sense, and in particular need not be reflected in any traditional notion of "belief" of the decision maker. Ultimately, the issue is a technical one. What we require is that, for any constant act and set of agents, all constant allocations of that act are Pareto efficient (and that any Pareto efficient allocation is weakly dominated by a constant allocation). This requirement amounts to requiring that for any preference profile, all agents share some common supporting hyperplane at all points on the ray of equal coordinates. The easiest and most intuitive way of doing this is fixing some prior ex-ante, and considering preferences which are risk averse (in the sense formerly defined) with respect to this probability.

Our aim in this study is to understand methods of aggregating preferences which reduce risk aversion. We imagine a set of agents who reside in a household and use some social welfare functional to optimally distribute resources. Samuelson (1956) observed that such optimization leads to "ra-

tional" behavior in the aggregate. We ask when such household behavior is less risk averse than the behavior of each individual in the household.

To this end, we discuss a comparative notion of risk aversion. For $c \in \mathbb{R}_+$, we abuse notation and identify c with the constant act whose outcome in every state is c. Let R' and R be two risk averse preference relations. As in Yaari (1969), we say that R **is more risk averse than** R' if for all $c \geq 0$, $\{x : x \, R \, c\} \subseteq \{x : x \, R' \, c\}$. Every uncertain prospect which is preferred to c by R is also preferred to c by R'.[9]

For a preference $R \in \mathcal{R}$, the **certainty equivalent** $ce_R : X \to \mathbb{R}$ is defined by

$$ce_R(x) = \inf \{c : c \, R \, x\}.$$

It is the value of the unique constant act which is indifferent to x; that is, by monotonicity and continuity, $ce_R(x) \, I \, x$.[10] Critically for us, for a given R, $ce_R : X \to \mathbb{R}$ is a continuous utility representation of R.

For an allocation $\mathbf{x} \in X^N$ and preference profile $\mathbf{R} \in \mathcal{R}^N$, $\mathbf{ce_R}(\mathbf{x}) = (ce_{R_1}(x_1), ..., ce_{R_n}(x))$.

> **Remark:** *For two preferences $R, R' \in \mathcal{R}$, R is more risk averse than R' if and only if for all $x \in X$, $ce_R(x) \leq ce_{R'}(x)$.*

A **domain** $\mathcal{D}$ is a nonempty subset of $\mathcal{R}^N$. A social welfare functional is a mapping which carries $\mathcal{D}$ into binary relations over X^N, the space of allocations. Formally, we denote the set of binary relations over X^N by $\mathcal{R}_N$. Then a **social welfare functional** is a function $\succeq^0 : \mathcal{D} \to \mathcal{R}_N$. We

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[9]Epstein (1999) and Ghirardato and Marinacci (2002) provide modern adaptations of this comparative notion to general settings of Knightian uncertainty. Their definitions differ as to the benchmark of "uncertainty neutral" acts, but coincide with Yaari's when uncertainty neutral acts are taken to be the constant acts.

[10]We write $x \, I \, y$ for $x \, R \, y$ and $y \, R \, x$.

write $\succeq^0(\mathbf{R})$ for the binary relation over allocations obtained when individual preferences are $\mathbf{R} = (R_i)_{i \in N}$.

Our paper is about risk sharing, so we need to formally define what we mean by this. It is the standard notion of risk sharing, in an environment of possibly non-expected-utility maximizers. Agents in a household share risk when they allocate a risk amongst themselves (see e.g. Epstein (2001) and Rigotti and Shannon (2005)).

The individuals in N are all members of a household. Household members entertain different attitudes toward risk. We imagine that the household uses a SWF to allocate an aggregate bundle x among its members. That is, given individual preferences $\mathbf{R}$, they maximize $\succeq^0 (\mathbf{R})$ across $\left\{ \mathbf{x} \in X^N : \sum_{i \in N} x_i \leq x \right\}$. Under our continuity assumptions, this risk-sharing maximization problem is well-defined. Risk sharing generates a well-defined household preference over acts: a "household preference." This aggregation results from a well-known aggregation result in classical demand theory (see Samuelson (1956) and Chipman and Moore (1979)).

The aggregate household preferences are derived from $\succeq^0(\mathbf{R})$: Household preferences over acts are given by $x \succeq^h (\mathbf{R}) y$ if and only if for all $\mathbf{y} \in X^N$ such that $\sum_{i \in N} y_i \leq y$, there exists $\mathbf{x} \in X^N$ such that $\sum_{i \in N} x_i \leq x$ and $\mathbf{x} \succeq^0 (\mathbf{R}) \mathbf{y}$. This binary relation is the **household preference**. In particular, $x \succeq^h (\mathbf{R}) y$ whenever the maximal allocation of x according to $\succeq^0 (\mathbf{R})$ is better than *every* allocation of y.

In this paper, we will be concerned with the following Pareto property for a SWF. It is a natural and standard definition.

> **Pareto:** *For all $\mathbf{R} \in \mathcal{D}$ and all $\mathbf{x}, \mathbf{y} \in X^N$, if $x_i R_i y_i$ for all $i \in N$, then $\mathbf{x} \succeq^0 (\mathbf{R}) \mathbf{y}$ (with strict preference if all individual preferences are strict).*

The following simple proposition illustrates that the household preference is risk averse. The intuition for the result is similar to results appearing in

Billot, Chateauneuf, Gilboa, and Tallon (2000), Dana (2002) and Rigotti, Shannon, and Strzalecki (2008) in that it relies on all agents having a common supporting hyperplane at rays of equal coordinates.

Proposition 1: *Suppose that $\succeq^0$ satisfies Pareto. Then for all $x \in X$,* $(\pi \cdot x) \succeq^h (\mathbf{R}) \, x$.

Example 2: A classical domain of preferences is the domain of risk averse expected utility profiles. We denote this domain by $\mathcal{EU}$. Formally, $\mathbf{R} \in \mathcal{EU}$ if for all $i \in N$, there exists $u_i : \mathbb{R}_+ \to \mathbb{R}$ which is strictly increasing, concave, and continuous for which for all $x, y \in X$, $x \, R_i \, y$ if and only if

$$\sum_{\omega \in \Omega} \pi\left(\omega\right) u_i\left(x\left(\omega\right)\right) \geq \sum_{\omega \in \Omega} \pi\left(\omega\right) u_i\left(y\left(\omega\right)\right).$$

A standard example of a social welfare functional $\succeq^0 \colon \mathcal{EU} \to \mathcal{R}_N$ used in the theory of risk sharing (for example, Borch (1962) and Wilson (1968)) is the **utilitarian rule**. For any $\mathbf{R} \in \mathcal{EU}$, there exists for all $i \in N$ a unique $u_{R_i} : \mathbb{R}_+ \to \mathbb{R}$ which represents R_i and which is normalized so that $u_{R_i}(0) = 0$ and $u_{R_i}(1) = 1$. We then require $\mathbf{x} \succeq^0 (\mathbf{R}) \, \mathbf{y}$ if and only if

$$\sum_{i \in N} \sum_{\omega \in \Omega} \pi\left(\omega\right) u_{R_i}\left(x_i\left(\omega\right)\right) \geq \sum_{i \in N} \sum_{\omega \in \Omega} \pi\left(\omega\right) u_{R_i}\left(y_i\left(\omega\right)\right).$$

For $c \in \mathbb{R}_+$, the c-simplex $\Delta_c = \left\{ \mathbf{d} \in \mathbb{R}_+^N : \sum_{i \in N} d_i = c \right\}$ is the set of nonnegative vectors summing to c. The notation $(c_i, 0_{-i})$ refers to the element of X^N whose ith coordinate is c, and all remaining coordinates are 0.

Formally, define an **anyone can take all (ACTA) rule** as a SWF for which there exists some strictly monotonic, continuous $W : \mathbb{R}_+^N \to \mathbb{R}$ for which for all $c \in \mathbb{R}_+$, for all $i \in N$,

$$(c_i, 0_{-i}) \in \arg \max_{\mathbf{d} \in \Delta_c} W\left(\mathbf{d}\right)$$

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such that for all $\mathbf{x}, \mathbf{y} \in X^N$ and all $\mathbf{R} \in \mathcal{R}^N$, $\mathbf{x} \succeq^0 (\mathbf{R}) \mathbf{y}$ if and only if $W(\mathbf{ce_R}(\mathbf{x})) \geq W(\mathbf{ce_R}(\mathbf{y}))$. The terminology ACTA refers to the fact that, for any riskless amount c, an ACTA rule finds it optimal to distribute *all* of c to some individual in the household–but the rule is completely indifferent as to *which* individual is to receive c.

Examples of W functions generating ACTA rules are the max function and the sum (utilitarian) function as discussed in Examples 2 and 5. Other examples include the functions $W(\mathbf{d}) = \left(\sum_{i \in N} d_i^p \right)^{1/p}$ for $p \geq 1$.

> **Remark:** *Because the ACTA rules recommend that giving everything to one individual is optimal in situations with no risk, one might confuse them with dictatorial rules, and hence view our following characterization as an impossibility. This is not correct for two reasons. For one, the fact that giving everything to one individual is optimal does not preclude the optimality of equal division, or any other division of resources. Secondly, nearly any social welfare existing in the literature which is not of the maxmin form will recommend giving everything to one agent in some situations. Simply because the situations in which ACTA rules make such recommendations are situations involving no risk does not mean we have an impossibility result.*

Note that, while we have restricted elements of $\mathcal{R}$ significantly, elements of $\mathcal{R}_N$ obey no restrictions whatsoever (we will later make assumptions on these elements in the form of axioms). This is because elements of $\mathcal{R}$ are understood to be descriptive, whereas elements of $\mathcal{R}_N$ are normative recommendations for a society.

3   Results

We proceed to describe the four axioms. The axioms will be equivalent to a certain class of SWF's. Coupled with quasiconcavity over riskless allocations, we characterize the SWF which is represented as the sum of certainty equivalents.

Our first axiom states that household preferences over allocations should be minimally "rational."

> **Rationality:** *For all $\mathbf{R} \in \mathcal{D}$, $\succeq^0(\mathbf{R})$ is continuous, complete, and transitive.*

Our second axiom is natural and requires that the SWF comply with the (weak) Pareto property discussed previously.

Our next axiom is the first that deals specifically with the interpretation of risk. It requires that in ranking profiles of certain (risk-free) prospects, the social welfare functional should ignore risk attitudes. Constant acts can be identified with monetary payoffs, and individual preferences are monotonic, so all individual preferences coincide over constant acts: more is better. The invariance axiom says that, when comparing constant allocations, $\succeq^0$ should not depend on individual risk preferences. These preferences do not differ in the comparison of constant acts anyhow.

> **Invariance to risk attitudes for constant acts:** *For all $\mathbf{R}, \mathbf{R}' \in \mathcal{D}$ and all constant $\mathbf{c}, \mathbf{d} \in X^N$, $\mathbf{c} \succeq^0(\mathbf{R}) \, \mathbf{d} \Longleftrightarrow \mathbf{c} \succeq^0(\mathbf{R}') \, \mathbf{d}$.*

We often refer to the axiom simply as *invariance*. Mathematically, together with the Pareto property, the axiom allows us to work with a ranking over vectors of certainty equivalents (defined below). This ranking over vectors of certainty equivalents is independent of the preference profile in question.

Remark: *It is often the case for expected utility preferences that the von Neumann-Morgenstern utility index is somehow understood as a "cardinal" measure of preference. It is true that the concavity of this index relates to risk aversion and comparative risk aversion. But this cardinal structure need not have any meaning when making ethical comparisons across agents. For example, there are no choice-based grounds for interpreting the curvature of the utility index as being related to marginal utility of income, when marginal utility of income is understood as we define it above.[11] We argue here that it is reasonable to ignore risk attitudes altogether when ranking constant prospects: when one decides on a "fair" way to split a certain dollar, it is rare that a discussion of attitudes toward risk would come into play. Nevertheless, as an anonymous referee has noted, it is certainly plausible to identify risk attitudes with marginal utility of income, and such an identification cannot be tested in our model. With such an interpretation, our invariance axiom can be seen as quite strong–it disallows us from considering marginal utility of income.*

Proposition 3: *A social welfare functional on $\mathcal{D}$ satisfies rationality, Pareto, and invariance if and only if there exists a strictly monotonic and continuous function $W : \mathbb{R}_+^N \to \mathbb{R}$ for which for all $\mathbf{R} \in \mathcal{D}$ and all $\mathbf{x}, \mathbf{y} \in X^N$,*

$$\mathbf{x} \succeq^0 (\mathbf{R}) \, \mathbf{y} \iff W\left(\mathbf{ce_R}\left(\mathbf{x}\right)\right) \geq W\left(\mathbf{ce_R}\left(\mathbf{y}\right)\right).^{12}$$

We are now ready to state our next axiom.

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[11]Marginal utilities would have to be elicited by a utility differences model.

[12]We say a function $W : \mathbb{R}_+^N \to \mathbb{R}$ is strictly monotonic if $\mathbf{x}, \mathbf{y} \in \mathbb{R}_+^N$ and $\mathbf{x} \geq \mathbf{y}$ implies $W\left(\mathbf{x}\right) \geq W\left(\mathbf{y}\right)$, and $\mathbf{x} \gg \mathbf{y}$ ($x_i > y_i$ for all $i \in N$) implies $W\left(\mathbf{x}\right) > W\left(\mathbf{y}\right)$.

Reduction of risk aversion: *For all* $\mathbf{R} \in \mathcal{D}$, $\succeq^h(\mathbf{R})$ *is less risk averse than* R_i *for all* $i \in N$.

Reduction of risk aversion is a way of capturing, in a fixed population framework, the notion that households of individuals should tend to be risk neutral. It also makes sense as a strategic property: in inter-household bargaining, the incentive will usually exist to be as risk neutral as possible. For example, the Nash bargaining solution tends to give more to less risk averse households: see Kihlstrom, Roth, and Schmeidler (1981); Rubinstein, Safra, and Thomson (1992).

Remark: *Reduction of risk aversion is much stronger than the requirement that the household preference be locally less risk averse than each individual agent at optimal consumption. Indeed, the requirement of less local risk aversion would be satisfied for essentially any social welfare functional satisfying the Pareto property.*

Example 4: Suppose $\mathcal{D} = \mathcal{EU}$ and consider the utilitarian rule defined in Example 2. In general, for any $\mathbf{R} \in \mathcal{EU}$, the household preference $\succeq^h(\mathbf{R})$ is expected utility with von Neumann Morgenstern utility index given by

$$u\left(x\right) = \sup_{\sum_{i \in N} x_i = x} \sum_{i \in N} u_{R_i}\left(x_i\right).$$

The function u is referred to as the "sup-convolution" of the functions u_{R_i}. It is easily verified that the resulting $\succeq^h$ need not have any relation to the individual preferences in terms of attitudes toward risk. For example, if we consider two identical agents with piecewise linear (in two pieces) utility indices with a kink at point x, then the sup-convolution will be piecewise linear with a kink at point $2x$. These functions cannot possibly be concave transformations of each other. Therefore, this social welfare functional does not satisfy reduction of risk aversion.

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(a) Preference R_1 is less risk averse than R_2.

(b) Household preference must contain lower envelope of $\bar{U}_1 - \bar{U}_1$ and $\bar{U}_2 - \bar{U}_2$.
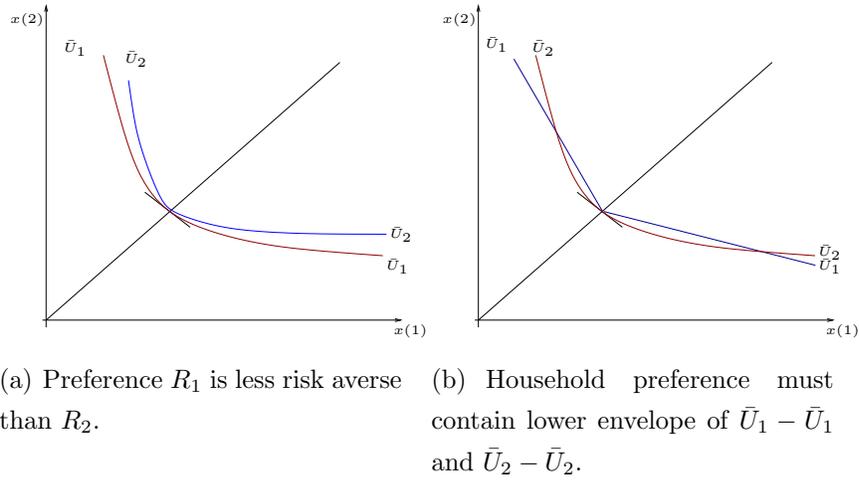
Figure 1: Comparisons in risk aversion.

Figure 1 presents the problem geometrically. One preference R_1 is less risk averse than R_2 if, when we compare their upper contour sets through any riskless act, the upper contour set of R_1 contains that of R_2. Figure 1(a) illustrates how the preferences having the $\bar{U}_1 - \bar{U}_1$ indifference curve are less risk averse than those having the $\bar{U}_2 - \bar{U}_2$ curve. The tangent line in the figure indicates the prior π. In Figure 1(a), any household preference satisfying the reduction of risk aversion axiom will need to have indifference curves "below" $\bar{U}_1 - \bar{U}_1$.

Figure 1(b) presents a case where the preferences do not have comparable risk aversion. In the figure, the most risk averse household preference which is less risk averse than each individual agent is given by the lower envelope of the two indifference curves (i.e. by the curve which goes from $\bar{U}_1$ to the intersection of the two indifference curves, then coincides with $\bar{U}_2 - \bar{U}_2$ until the second intersection, then coincides with $\bar{U}_1 - \bar{U}_1$). Note that this household preference is not convex, and any household preference satisfying reduction of risk aversion must have indifference curves below this lower envelope of $\bar{U}_1 - \bar{U}_1$ and $\bar{U}_2 - \bar{U}_2$.

Our aim from this point on is to characterize those functions W which reduce risk aversion.

Example 5: One of the simplest examples of a W which reduces risk aversion is given by

$$W\left(d_{1},...,d_{n}\right) = \max_{i \in N}\left\{d_{i}\right\}.$$

This function illustrates some of the properties of the reduction of risk aversion. The household rule generated by this function reduces risk aversion, but the function W constitutes a very unfair rule. Moreover, the induced household preferences are typically not convex, even when the individual preferences are convex. If we denote by $U_{i}\left(c\right)$ the upper contour set of R_{i} at c, then it is easily verified that $U^{*}\left(c\right) = \bigcup_{i \in N} U_{i}\left(c\right)$ (this also verifies that this household preference is the most risk averse preference which is less risk averse than each individual preference in the household).

The following result is our main characterization theorem. It tells us that under our axioms, a social welfare functional reduces risk aversion if and only if it is associated with a function W which is maximized on any c-simplex at the vertices–that is, it is an ACTA rule. The class of W corresponding to ACTA rules are illustrated in Figure 2.

The theorem holds on very general domains; however, in the statement of our theorem, we require that the domain include all expected utility profiles. A close inspection of the proof establishes that we only need to assume that our domain includes, for every individual, a profile for which that individual is risk neutral, and all remaining individuals have expected utility preferences whose utility index satisfies the Inada conditions.

Theorem 6: *Suppose that $\mathcal{EU} \subseteq \mathcal{D}$. A social welfare functional satisfies rationality, Pareto, invariance, and reduction of risk aversion on $\mathcal{D}$ if and only if it is an ACTA rule.*
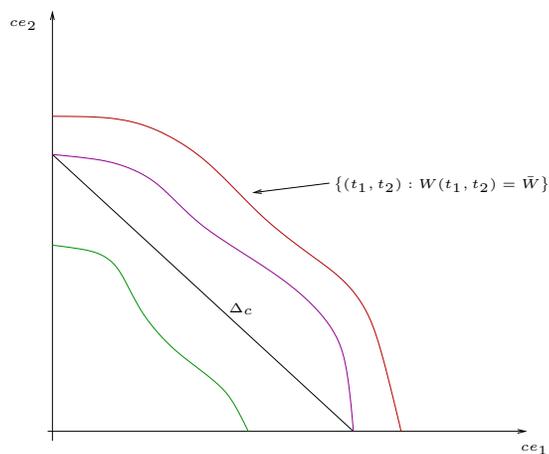
Figure 2: Level curves of W in the hypotheses of Theorem 6.
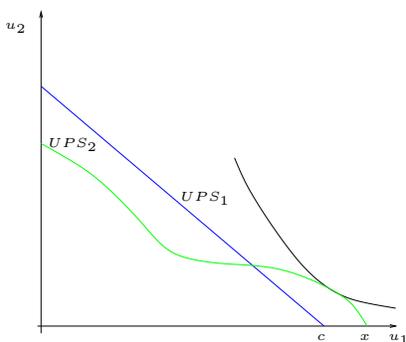
Figure 3: Some intuition for Theorem 6.

A simple incomplete intuition for the proof of Theorem 6 exists. Any aggregate bundle generates a utility possibility set, where utilities are given by certainty equivalents. The definition of our household preference suggests that for any profile of individual preferences, to rank aggregate bundles, we rank utility possibility sets by their best element according to some function W–see Proposition 3. The Pareto and invariance conditions are used to guarantee that this same social welfare function W is used for all profiles of individual preferences.

Now consider a constant aggregate bundle c. For the certainty equivalent representation, the utility possibility set of c is a simplex. We represent the situation in Figure 3; the utility possibility set of c is represented as UPS_1 in blue. Suppose there is now another bundle x which some individual i (in the figure individual 1) prefers to the constant bundle. This means that the utility possibility set for bundle x, UPS_2, extends beyond the simplex on i's axis. By the reduction in risk aversion axiom, since 1 prefers x over c, so must the household. In fact, even if we choose the remaining individuals' utility functions so that they get very little utility from x, the household should still prefer x to c. This means that there is some list of individual utilities corresponding to some allocation of x which lies outside of the simplex, and which is ranked higher than every element of the simplex by W. Because the only list of individual utilities lying outside of the simplex are ones in which individual i gets "almost all" of the utility, it seems reasonable to conclude that W corresponds to an ACTA rule.

Our formal proof does not follow this construction, and instead uses a "dual" representation, based on demand functions.

Figure 2 shows the level curves of a W corresponding to an ACTA rule. It should be clear that requiring W to be quasiconcave pins down the sum of certainty equivalents. The following axiom contains the quasiconcavity restriction. It has the interpretation that in an "divide the dollar" environment, without risk, social preferences should be "fair."

Quasiconcavity: *Let* $\mathbf{c}$ *and* $\mathbf{d}$ *be constant allocations. Let* $\mathbf{R} \in \mathcal{R}^N$. *Suppose* $\mathbf{c} \succeq^0 (\mathbf{R}) \mathbf{d}$. *Then for all* $\alpha \in [0, 1]$, $\alpha \mathbf{c} + (1 - \alpha) \mathbf{d} \succeq^0 (\mathbf{R}) \mathbf{d}$.

Corollary 7: *Suppose that* $\mathcal{EU} \subseteq \mathcal{D}$. *A social welfare functional* $\succeq^0$ *satisfies rationality, Pareto, invariance, reduction of risk aversion, and*

quasiconcavity if and only if for all $\mathbf{R} \in \mathcal{D}$ and all $\mathbf{x}, \mathbf{y} \in X^N$,

$$\mathbf{x} \succeq^0 (\mathbf{R}) \, \mathbf{y} \iff \sum_{i \in N} ce_{R_i}(x_i) \geq \sum_{i \in N} ce_{R_i}(y_i).$$

Corollary 7 tells us that the only "fair" SWF to reduce risk aversion is the one which ranks allocations according to the sum of its certainty equivalents. When allocating a constant act by the sum of certainty equivalents, all constant allocations are socially optimal.

> **Remark:** *We could replace the hypothesis of quasiconcavity by Schur-concavity, delivering the same result.*

4 Application: homothetic preferences

Theorem 6 gives a family of functions that reduce risk aversion. We singled out the sum of certainty equivalents based on quasiconcavity. Here we provide another justification, one that holds for profiles of homothetic preferences. The model of **?** is especially suitable for the results of this section.

The maximum function in Example 5 is the most risk averse preference which is less risk averse than all individual preferences. As we remarked, this rule may in general induce non-convex household preferences even when all individual preferences are convex: see Figure 1(b).

Here we study the most risk averse *convex* preference which is less risk averse than all individual preferences for convex individual preferences. We show that, for profiles of convex and homothetic preferences, the sum of certainty equivalents gives the most risk averse convex preference that is less risk averse than members' preferences.

In Figure 1(b), the most risk averse convex household preference relation is given by the convex hull of the two upper contour sets. If we denote the

upper contour set of agent i's preference at c as $U_i(c)$, the upper contour set of the household preference at c is

$$\overline{\text{co} \bigcup_{i \in N} U_i(c)},$$

the closed convex hull of the union of the individual upper contour sets. We shall prove this below.

Say a preference $R \in \mathcal{R}$ is **homothetic** if for all $x, y \in X$ and all $\alpha > 0$, $x \, R \, y \implies (\alpha x) \, R \, (\alpha y)$. Denote the set of homothetic and convex preferences by $\mathcal{H}$.

> **Theorem 8:** *Suppose that* $\mathbf{R} \in \mathcal{H}^N \cap \mathcal{R}^N$. *Consider the SWF represented by the sum of certainty equivalents. Then the household preference* $\succeq^h (\mathbf{R})$ *is homothetic, and is the most risk averse convex preference which is less risk averse than* R_i *for all* $i \in N$.

The theorem demonstrates that at any constant act, the upper contour set of the household preference is the closed convex hull of the union of the individual upper contour sets. This means that the household preference generated by the sum of certainty equivalents is both tractable and geometrically simple.[13]

The proof of Theorem 8 exploits the fact that every profile of common prior homothetic preferences has, for each agent, a representation as: $ce_{R_i}(x) = \inf_{y \in C_i} x \cdot y$, where the common prior π minimizes $\sum_{\omega \in \Omega} y(\omega)$ in C_i. In particular, this set C_i can be explicitly calculated as

$$C_i = \{ y : x \, R_i \, 1 \implies x \cdot y \geq_i 1 \}.$$

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[13]There is a similar result in the theory of international trade, on the maximization of profits under constant returns to scale and more than one industry. Lerner (1934) and Chipman (1966) present a "diagrammatic" argument.

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Using this representation, it is easy to explicitly calculate household preference: it is given by $U_{\succeq^h(\mathbf{R})}(x) = \inf_{y \in \bigcap_{i \in N} C_i} x \cdot y$, verifying the tractability of the sum of individual certainty equivalents as an instrument of applied analysis.

Example 9: "Multiple priors": Suppose that for all $i \in N$, there exists some Π_i for which $\pi \in \Pi_i$; and that for all $i \in N$, $ce_{R_i}(x) = \min_{\pi^* \in \Pi_i \subseteq \Delta(\Omega)} \pi^* \cdot x$. Then $x \succeq^h(\mathbf{R}) y \iff \min_{\pi^* \in \bigcap \Pi_i + \mathbb{R}_+^\Omega} \pi^* \cdot x \geq \min_{\pi^* \in \bigcap \Pi_i + \mathbb{R}_+^\Omega} \pi^* \cdot y$.[14]

Example 10: CRRA expected utility maximizers: Suppose that for all $i \in N$, $ce_{R_i}(x) = \left(\int_\Omega [x(\omega)]^{\rho_i} \, d\pi(\omega) \right)^{\frac{1}{\rho_i}}$ for $\rho_i \in [0,1]$. Then

$$x \succeq^h(\mathbf{R}) y$$
$$\iff \left(\int_\Omega [x(\omega)]^{\max_{i \in N} \rho_i} \, d\pi(\omega) \right)^{1/\max_{i \in N} \rho_i} \geq \left(\int_\Omega [y(\omega)]^{\max_{i \in N} \rho_i} \, d\pi(\omega) \right)^{1/\max_{i \in N} \rho_i}.$$

5 Conclusion

We study household preferences in the context of sharing risk and uncertainty. We are especially interested in the sum of certainty equivalents as a criterion for choosing and allocating risky bundles. We identify a crucial property behind this criterion: that aggregate preferences are less risk averse than the members' individual preferences.

Arguments for reduction in risk aversion are familiar in economics, and appear as early as in Samuelson (1964), Vickrey (1964) and Arrow and Lind (1970). These arguments are normative: a collective *should* behave in a less risk averse way. The arguments roughly say that less cautious collectives may reap the benefits of larger expected gains, and mitigate the risks by

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[14]For two sets A and B, $A + B = \{x + y : x \in A, y \in B\}$.

risk sharing. The idea that a household should be less risk averse is also strategically motivated. For example, often in strategic interactions, all else equal, an agent who is less risk averse will fare better according to all preferences. This is the case in Nash bargaining (Rubinstein, Safra, and Thomson (1992)), as well as in many other game theoretic models of bargaining. So, by appropriately sharing risk, a household seeks to become more "competitive."

Our result says much more than that ACTA rules yield a *local* reduction in risk aversion. The local statement would be that the household is locally less risk averse than each individual at each individual's optimal consumption. Indeed, this would be true for *any* rule satisfying the Pareto property. Our result is much stronger: ACTA rules generate a household which is less risk averse than each of its members *at every level of consumption.*

We introduce two additional axioms: the Pareto criterion and that certain (sure) acts should be compared without regard for risk preferences. From the normative perspective, the Pareto criterion is obviously desirable, and the invariance axiom should be appealing. invariance may not be appealing in a descriptive setting, in which any one agent can force a "breakdown" of negotiations; then risk attitudes play a role even when the "optimal" choices feature no risk (Rubinstein, Safra, and Thomson (1992) explain how attitudes toward risk are important in a Nash bargaining context).

We characterize the household SWF's which respect the Pareto criterion, compare certain acts without regard for risk preferences, and which reduce risk aversion. The results single out the sum of certainty equivalents as the unique member of this class which is quasiconcave over certain allocations.

Quasiconcavity, in turn, is a basic fairness requirement.[15]

Our results hold when we restrict the domain to expected utility preferences. this should be clear from the statements and proofs in Section 3. The results are also novel as results about $\mathcal{EU}$; but no additional insights or simplifications are gained from presenting them as results about $\mathcal{EU}$. We emphasize; however, that our proof require that any domain include all expected utility preferences.[16]

Our paper thus provides a justification for using the sum of certainty equivalents as a guide in allocating risky prospects. The existing justification (Myerson, 2004) says that any allocation which does not maximize the sum of certainty equivalents could be improved upon by an allocation that each individual agent would be willing to accept more for. This is a simple consequence of interpreting certainty equivalents as willingness to accept. Our justification is entirely different, and depends on the desirability of the collective being less risk averse than household members.

6   Appendix: Proofs of results

PROOF (PROOF OF PROPOSITION 1): Let $x \in X$ and let $\mathbf{x} \in X^N$ such that $\sum_{i \in N} x_i = x$. Then, for all $i \in N$, $(\pi \cdot x_i)\, R_i\, x_i$. Clearly, $\sum_{i \in N} \pi \cdot x_i = \pi \cdot \sum_{i \in N} x_i = \pi \cdot x$. By Pareto, $(\pi \cdot x_1, ..., \pi \cdot x_n) \succeq^0 (\mathbf{R})\, \mathbf{x}$. Consequently, by definition of $\succeq^h (\mathbf{R})$, $\pi \cdot x \succeq^h (\mathbf{R})\, x$. $\square$

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[15]Gorman (1959) argues that actual collectives may use a convex W: he believes that utility profiles that are not very unequal may be inherently stable. Any small advantage obtained by a group of agents will result in a political advantage, which will then reinforce the initially small advantage. The resulting collective will behave as if it used a convex social choice function.

[16]Actually, a close reading of our proof indicates that we only require that the domain include all risk neutral preferences, and at least one expected utility preference satisfying the Inada conditions.

PROOF (PROOF OF PROPOSITION 3): Let $\mathbf{R}' \in \mathcal{D}$. Define $W : \mathbb{R}_+^N \to \mathbb{R}$ so that $W(\mathbf{d}) \geq W(\mathbf{c})$ if and only if

$$\mathbf{d} \succeq^0 (\mathbf{R}') \, \mathbf{c}.$$

Such a W exists as $\succeq^0 (\mathbf{R}')$ is continuous and satisfies rationality, by (Debreu, 1964). W is strictly monotonic by the Pareto property. Now, let $\mathbf{R} \in \mathcal{D}$ be arbitrary. Let $\mathbf{x}, \mathbf{y} \in X^N$. Note that by Pareto, $\mathbf{ce_R}(\mathbf{x}) \sim^0 (\mathbf{R}) \, \mathbf{x}$ and $\mathbf{ce_R}(\mathbf{y}) \sim^0 (\mathbf{R}) \, \mathbf{y}$. Consequently, by rationality, $\mathbf{x} \succeq^0 (\mathbf{R}) \, \mathbf{y}$ if and only if $\mathbf{ce_R}(\mathbf{x}) \succeq^0 (\mathbf{R}) \, \mathbf{ce_R}(\mathbf{y})$. By invariance $\mathbf{ce_R}(\mathbf{x}) \succeq^0 (\mathbf{R}) \, \mathbf{ce_R}(\mathbf{y})$ if and only if $\mathbf{ce_R}(\mathbf{x}) \succeq^0 (\mathbf{R}') \, \mathbf{ce_R}(\mathbf{y})$. By definition of W, $\mathbf{ce_R}(\mathbf{x}) \succeq^0 (\mathbf{R}') \, \mathbf{ce_R}(\mathbf{y})$ if and only if $W(\mathbf{ce_R}(\mathbf{x})) \geq W(\mathbf{ce_R}(\mathbf{y}))$. $\qquad\square$

PROOF (PROOF OF THEOREM 6): Throughout, we use some abuse of notation between constant acts $c \in X$ and the value that constant act takes.

First, suppose that a social welfare functional $\succeq^0$ is an ACTA rule, associated with some function W satisfying the appropriate hypotheses. We shall prove that it satisfies the reduction of risk aversion axiom. That it satisfies the remaining axioms is immediate. Let $\mathbf{R} \in \mathcal{D}$. Let c be a constant act. We wish to show that for all $i \in N$, $\{x : x \, R_i \, c\} \subseteq \{x : x \succeq^h (\mathbf{R}) \, c\}$.

As a first step, we show that for all $i \in N$, $(c_i, 0_{-i}) \in \arg\max_{\mathbf{x} \in A_c} W(\mathbf{ce_R}(\mathbf{x}))$, where A_c is the set of allocations of the constant act c. That is, $A_c = \{\mathbf{y} \in X^N : \sum_{i \in N} y_i = c\}$.

Let $\mathbf{y} \in A_c$. For all $i \in N$, $(\pi \cdot y_i) \, R_i \, y_i$, and $(\pi \cdot y_1, ..., \pi \cdot y_n) \in \Delta_c$. So, for any $\mathbf{y} \in A_c$ there is $\mathbf{d} \in \Delta_c$ such that $W(\mathbf{d}) \geq W(\mathbf{ce_R}(\mathbf{x}))$. Continuity of W and compactness of Δ_c implies there exists an optimal allocation for W in Δ_c. Hence we may conclude that this optimal allocation is constant, and call it $\mathbf{d}^*$; since $\mathbf{d}^*$ is constant, with a slight abuse of notation, we write $\mathbf{d}^* \in \Delta_c$.

By the hypothesis on W, $W(c_i, 0_{-i}) \geq W(\mathbf{d}^*)$; we therefore establish that

$(c, 0_{-i}) \in \arg\max_{A_c} W$.

Now, let $x \in X$ and suppose that $x R_i c$. So $ce_{R_i}(x) \geq c$. Then for all $\mathbf{y} \in X^N$ for which $\sum_{i \in N} y_i = c$,

$$W(ce_{R_i}(x), 0_{-i}) \geq W(c_i, 0_{-i}) \geq W(y_1, \ldots, y_n).$$

Therefore, for every allocation $\mathbf{y}$ of c, $(x_i, 0_{-i}) \succeq^0(\mathbf{R}) \mathbf{y}$. By definition of $\succeq^h(\mathbf{R})$, $x \succeq^h(\mathbf{R}) c$.

Conversely, suppose that $\succeq^0$ satisfies the axioms. W exists from Proposition 3; we will show that the vertices of every simplex maximize W on the simplex.

Define $\Delta_{++}(\Omega) = \left\{ q \in \mathbb{R}_{++}^\Omega : \sum_\omega q(\omega) = 1 \right\}$ (the set of full support probability measures). We shall consider a profile $\mathbf{R} \in \mathcal{EU}$. Fix an arbitrary $j \in N$. Let R_j be defined by

$$ce_{R_j}(x) = \pi \cdot x$$

and for all $i \neq j$, choose some strictly increasing, concave, and differentiable $u_i : \mathbb{R}_+ \to \mathbb{R}$ for which $\lim_{x \to 0^+} u_i'(x) = +\infty$ and $\lim_{x \to +\infty} u_i'(x) = 0$ (the Inada conditions) and define R_i by

$$x R_i y \iff \sum_\omega \pi(\omega) u_i(x(\omega)) \geq \sum_\omega \pi(\omega) u_i(y(\omega)).$$

Note that $\mathbf{R} \in \mathcal{EU}$. Importantly for what follows, each R_i is a convex preference relation.

By Proposition 1, $\succeq^h(\mathbf{R})$ is risk averse. We shall prove that, by reduction of risk aversion, $\succeq^h(\mathbf{R}) = R_j$. To see this, let $c \in X$ be a constant act. Then as $\succeq^h(\mathbf{R})$ is risk averse, if $x \succeq^h(\mathbf{R}) c$, then $\pi \cdot x \geq c$; consequently $x R_j c$ by the definition of R_j. Hence, $\left\{ x : x \succeq^h(\mathbf{R}) c \right\} \subseteq \{x : x R_j c\}$. By reduction of risk aversion, we obtain that $\{x : x R_j c\} = \left\{ x : x \succeq^h(\mathbf{R}) c \right\}$. This implies that $ce_{R_j} = ce_{\succeq^h(\mathbf{R})}$; thus $R_j = \succeq^h(\mathbf{R})$.

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For each $i \in N$, define the **indirect utility function** $v_{R_i} : \Delta_{++}(\Omega) \times \mathbb{R}_+$ by

$$v_{R_i}(q, m) = \max_{q \cdot x \leq m} ce_{R_i}(x).$$

Define $U_{\succeq^h(\mathbf{R})} : X \to \mathbb{R}$ by

$$U_{\succeq^h(\mathbf{R})}(x) = \sup_{\sum x_i = x} W(\mathbf{ce_R}(\mathbf{x})).$$

Similarly, define the household indirect utility function by

$$V_{\succeq^h(\mathbf{R})}(q, m) = \max_{q \cdot x \leq m} U_{\succeq^h(\mathbf{R})}(x).$$

By Chipman and Moore (1979), Theorem 3.9,

$$V_{\succeq^h(\mathbf{R})}(q, m) = \max_{\mathbf{d} \in \Delta(N)} W\left((v_{R_i}(q, d_i m))_{i \in N}\right).$$

By the Maximum Theorem, the correspondence $\delta : \Delta(\Omega) \times \mathbb{R}_+ \rightrightarrows \Delta(N)$ defined by

$$\delta(q, m) = \arg \max_{\mathbf{d} \in \Delta(N)} W\left((v_{R_i}(q, d_i m))_{i \in N}\right)$$

is well-defined and upper semi-continuous. Define the demand correspondence $x_R(q, m)$ as those allocations which are R-maximal in the set $\{x : q \cdot x \leq m\}$.

By Chipman and Moore (1979), Corollary 3.5,

$$x_{\succeq^h(\mathbf{R})}(q, m) = \bigcup_{\mathbf{d} \in \delta(q, m)} \sum_{i \in N} x_{R_i}(q, d_i m).$$

Now, let $q \in \Delta_{++}(\Omega)$, $q \neq \pi$, $q \gg 0$. Since $\succeq^h(\mathbf{R})$ coincides with R_j, if $x \in x_{\succeq^h(\mathbf{R})}(q, m)$, then if $\frac{q(\omega)}{\pi(\omega)} > \frac{q(\omega')}{\pi(\omega')}$, $x(\omega) = 0$. Therefore, there exists ω for which $x(\omega) = 0$. Moreover, for all $i \neq j$, if $m > 0$, $x_{R_i}(q, m) \gg 0$. Consequently, we conclude that for all $\mathbf{d} \in \delta(q, m)$, $d_i = 0$ for $i \neq j$. By upper semicontinuity of δ, conclude that $(1_j, 0_{-j}) \in \delta(\pi, m)$.

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Note that for all $i \in N$ and all $\mathbf{d} \in \Delta(N)$, $v_{R_i}(\pi, d_i m) = d_i m$ (as v_{R_i} was defined using the certainty equivalent utility representation of R_i). Consequently $(v_{R_i}(\pi, d_i m))_{i \in N} \in \Delta_m$; and for any $u \in \Delta_m$, there exists $d \in \Delta(N)$ for which $(v_{R_i}(\pi, d_i m))_{i \in N} = u$. As $(1_j, 0_{-j}) \in \delta(q, m)$, we therefore conclude by definition of δ that $W(m_j, 0_{-j}) \geq W(u)$, for all $u \in \Delta_m$.

As j and m were arbitrary, the proof is complete. $\square$

PROOF (PROOF OF THEOREM 8): The following two lemmas are well-known, but we reproduce them here for completeness.

Lemma 11: *If $R \in \mathcal{H}$, then the function $ce_R : X \to \mathbb{R}$ is a utility representation for R which is homogeneous of degree one.*

PROOF: Recall
$$ce_R(x) = \inf\{c : c\,R\,x\}.$$

To see that the certainty equivalent is homogeneous, let $x \in X$ and $\alpha > 0$. Then

$$
\begin{aligned}
ce_R(\alpha x) &= \inf\{\alpha c : \alpha c\,R\,\alpha x\}\\
&= \alpha \inf\{c : \alpha c\,R\,\alpha x\}\\
&= \alpha \inf\{c : c\,R\,x\}\\
&= \alpha ce_R(x),
\end{aligned}
$$

where the second to last equality holds by homotheticity. $\square$

Lemma 12: *If u is monotone, homogeneous of degree one, and quasi-concave, then it is concave.*

PROOF: Let $x, y \in X$ and $\alpha \in [0, 1]$. Suppose without loss of generality that $u(y) \geq u(x)$. If $u(x) = 0$, then by monotonicity,

$$
\begin{aligned}
u(\alpha x + (1 - \alpha)y) &\geq u((1 - \alpha)y)\\
&= (1 - \alpha)u(y) = \alpha u(x) + (1 - \alpha)u(y),
\end{aligned}
$$

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verifying concavity. Otherwise, suppose $u(x) > 0$. Then by homogeneity, we know that $u\left(\frac{u(x)}{u(y)}y\right) = u(x)$. Let $\beta = \frac{\alpha u(x)}{\alpha u(x) + (1-\alpha)u(y)}$. By quasiconcavity, $u\left(\beta x + (1-\beta)\frac{u(x)}{u(y)}y\right) \geq u(x)$. But

$$\beta x + (1-\beta)\frac{u(x)}{u(y)}y = \left(\frac{\alpha u(x)}{\alpha u(x) + (1-\alpha)u(y)}\right)x + \left(\frac{(1-\alpha)u(x)}{\alpha u(x) + (1-\alpha)u(y)}\right)y.$$

Therefore, by homogeneity, $u\left(\beta x + (1-\beta)\frac{u(x)}{u(y)}y\right) = \frac{u(x)}{\alpha u(x) + (1-\alpha)u(y)}u(\alpha x + (1-\alpha)y)$. So, we have

$$\frac{u(x)}{\alpha u(x) + (1-\alpha)u(y)}u(\alpha x + (1-\alpha)y) \geq u(x),$$

which works out to $u(\alpha x + (1-\alpha)y) \geq \alpha u(x) + (1-\alpha)u(y)$ after canceling terms, verifying concavity. $\qquad\square$

Let $\mathbf{R} \in \mathcal{H}^N \cap \mathcal{R}$. By Lemmas 11 and 12, the certainty equivalent function $ce_{R_i} : X \to \mathbb{R}$ is homogeneous and concave. Moreover, for all constant acts c, $ce_{R_i}(c) = c$. Extend ce_{R_i} to all of $\mathbb{R}^\Omega$ by defining

$$ce'_{R_i}(x) = \begin{cases} ce_{R_i}(x) \text{ if } x \geq 0 \\ -\infty \text{ otherwise} \end{cases}.$$

The function ce'_{R_i} is concave, monotonic, and upper semicontinuous. Its conjugate, $\left(ce'_{R_i}\right)^* : \mathbb{R}^N \to \mathbb{R}$ is defined by

$$\left(ce'_{R_i}\right)^*(x) = \inf_{y \in \mathbb{R}^\Omega} x \cdot y - ce'_{R_i}(y).$$

It is well-known that this function is itself concave and that there is a nonempty, closed, convex, upper comprehensive[17] set $C_i \subseteq \mathbb{R}^N_+$ for which

$$\left(ce'_{R_i}\right)^*(x) = \begin{cases} 0 \text{ if } x \in C_i \\ -\infty \text{ otherwise.} \end{cases}.$$

---

[17]That is, if $x \in C$ and $y \geq x$, then $y \in C$.

Moreover,

$$ce'_{R_i}(x) = \inf_{y \in C_i} x \cdot y.$$

(See, Rockafellar (1970) Theorem 12.2, Theorem 13.2 and Corollary 13.2.1).

We claim that $\pi \in \bigcap_{i \in N} C_i$ and moreover that π lies on the boundary (has minimal sum) of each C_i. To see this, note that for each i and each constant act c, $ce'_{R_i}(c) = c$, so $1 = ce'_{R_i}(1) = \inf_{y \in C_i} 1 \cdot y = \inf_{y \in C_i} \sum_{\omega} y(\omega)$. Now, suppose that $\pi \notin C_i$ for some C_i. In particular, by a standard separation argument, there exists $x \in \mathbb{R}^N_+ \backslash \{0\}$ for which $\pi \cdot x < \inf_{y \in C_i} y \cdot x$. Let c be a real number for which $\pi \cdot x < c < \inf_{y \in C_i} y \cdot x = ce_{R_i}(x)$. But then $x \, R_i \, c$, while $\pi \cdot x < c$, contradicting the fact that R_i is risk averse.

Now consider the function defined on X for which

$$U_{\succeq^h(\mathbf{R})}(x) = \max_{\sum x_i = x} \sum ce_{R_i}(x_i).$$

Clearly, this function can also be defined on all of $\mathbb{R}^\Omega$, so that

$$U'_{\succeq^h(\mathbf{R})}(x) = \max_{\sum x_i = x} \sum ce'_{R_i}(x_i).$$

Moreover, it is easy to see, that since $U'_{\succeq^h(\mathbf{R})}$ takes infinite values outside of X, for $x \in X$,

$$U_{\succeq^h(\mathbf{R})}(x) = U'_{\succeq^h(\mathbf{R})}(x).$$

Finally, as $U'_{\succeq^h(\mathbf{R})}$ is the sup-convolution of the functions $\left(ce'_{R_i}\right)_{i \in N}$, we conclude that the conjugate

$$\left(U'_{\succeq^h(\mathbf{R})}\right)^*(x) = \inf_{y \in \mathbb{R}^N} x \cdot y - U'_{\succeq^h(\mathbf{R})}(y)$$

is given by

$$\left(U'_{\succeq^h(\mathbf{R})}\right)^*(x) = \sum_{i \in N}\left(ce'_{R_i}(x)\right)^*$$

$$= \left\{\begin{array}{l} 0 \text{ if } x \in \bigcap_{i \in N} C_i \\ -\infty \text{ otherwise.} \end{array}\right\}.$$

See Rockafellar (1970), Theorem 16.4 and Corollary 16.4.1. Consequently,

$$U'_{\succeq^h(\mathbf{R})}(x) = \inf_{y \in \bigcap_{i \in N} C_i} x \cdot y.$$

Importantly for these arguments, $\bigcap_{i \in N} C_i \neq \varnothing$, as each C_i is upper comprehensive and contains π. Hence, we conclude that household preference $\succeq^h(\mathbf{R})$ is represented by

$$x \succeq^h(\mathbf{R}) z \iff \inf_{y \in \bigcap_{i \in N} C_i} x \cdot y \geq \inf_{y \in \bigcap_N C_i} z \cdot y,$$

where for all $i \in N$,

$$x \ R_i \ z \iff \inf_{y \in C_i} x \cdot y \geq \inf_{y \in C_i} z \cdot y.$$

Clearly, then, $\succeq^h(\mathbf{R})$ is homothetic. To see that it is the most risk averse convex preference which is less risk averse than each individual preference, let c be a constant act. Note that $\pi \in \bigcap_{i \in N} C_i$ and also lies on the boundary of $\bigcap_{i \in N} C_i$ (it minimizes $\sum y(\omega)$ across $y \in \bigcap_{i \in N} C_i$). Consequently for any

constant act c,

$$c = c \sum_{\omega \in \Omega} \pi(\omega)$$

$$= c \inf_{y \in \bigcap_{i \in N} C_i} \sum_{\omega \in \Omega} y(\omega)$$

$$= \inf_{y \in \bigcap_{i \in N} C_i} c \cdot y$$

We will show that for any c,

$$\left\{ x : x \succeq^h (\mathbf{R}) \, c \right\} = \overline{\mathrm{co} \bigcup_{i \in N} \{x : x \, R_i \, c\}},$$

which will verify the result. So first, we show that for all $i \in N$, $\{x : x \, R_i \, c\} \subseteq \left\{ x : x \succeq^h (\mathbf{R}) \, c \right\}$. Note that $x \, R_i \, c$ implies that for all $y \in C_i$, $x \cdot y \geq c$ which implies that for all $y \in \bigcap_{i \in N} C_i$, $x \cdot y \geq c$, which implies that $x \succeq^h (\mathbf{R}) \, c$. We therefore know that

$$\overline{\mathrm{co} \bigcup_{i \in N} \{x : x \, R_i \, c\}} \subseteq \left\{ x : x \succeq^h (\mathbf{R}) \, c \right\}$$

as $\succeq^h (\mathbf{R})$ is upper semicontinuous and convex. Suppose now that there exists $w \in X$ such that $w \succeq^h (\mathbf{R}) \, c$, and for which $w \notin \overline{\mathrm{co} \bigcup_{i \in N} \{x : x \, R_i \, c\}}$. In particular, by a standard separation argument, there exists y for which, when normalized, $y \cdot w < c \leq y \cdot x$ for all i and all $x \, R_i \, c$. We claim that for all $i \in N$, $y \in C_i$; otherwise, there would exist a separating vector (again nonnegative and normalized) z for which $y \cdot z < c < \inf_{y' \in C_i} y' \cdot z$. But then $z \, R_i \, c$ and $y \cdot z < c$, contradicting $y \cdot x \geq c$ for all $x \, R_i \, c$. Consequently, $y \in \bigcap_{i \in N} C_i$. Therefore, $\inf_{y \in \bigcap_{i \in N} C_i} y \cdot w < c$, so that $c \succ^h (\mathbf{R}) \, w$, a contradiction. $\square$

References

Arrow, K. J., and R. C. Lind (1970): "Uncertainty and the Evaluation of Public Investment Decisions," *American Economic Review*, 60(3), 364–378.

Billot, A., A. Chateauneuf, I. Gilboa, and J.-M. Tallon (2000): "Sharing Beliefs: Between Agreeing and Disagreeing," *Econometrica*, 68(3), 685–694.

Borch, K. (1962): "Equilibrium in a Reinsurance Market," *Econometrica*, 30(3), 424–444.

Cerreia-Vioglio, S., F. Maccheroni, M. Marinacci, and L. Montrucchio (2008): "Uncertainty Averse Preferences," Carlo Alberto Notebooks 77, Collegio Carlo Alberto.

Chateauneuf, A., R.-A. Dana, and J.-M. Tallon (2000): "Optimal risk-sharing rules and equilibria with Choquet-expected-utility," *Journal of Mathematical Economics*, 34(2), 191 – 214.

Chipman, J. S. (1966): "A Survey of the Theory of International Trade: Part 3, The Modern Theory," *Econometrica*, 34(1), 18–76.

Chipman, J. S., and J. C. Moore (1979): "On social welfare functions and the aggregation of preferences," *Journal of Economic Theory*, 21(1), 111 – 139.

Dana, R.-A. (2002): "On Equilibria when Agents Have Multiple Priors," *Annals of Operations Research*, 114(1-4), 105–115.

Debreu, G. (1964): "Continuity Properties of Paretian Utility," *International Economic Review*, 5(3), 285–293.

DOW, J., AND S. R. D. C. WERLANG (1992): "Uncertainty Aversion, Risk Aversion, and the Optimal Choice of Portfolio," *Econometrica*, 60(1), 197–204.

EPSTEIN, L. G. (1999): "A Definition of Uncertainty Aversion," *Review of Economic Studies*, 66(3), 579–608.

——— (2001): "Sharing Ambiguity," *The American Economic Review*, 91(2), 45–50.

EPSTEIN, L. G., AND T. WANG (1994): "Intertemporal Asset Pricing under Knightian Uncertainty," *Econometrica*, 62(2), 283–322.

ERGIN, H., AND F. GUL (2008): "A Theory of Subjective Compound Lotteries," Forthcoming, Journal of Economic Theory.

GHIRARDATO, P., AND M. MARINACCI (2002): "Ambiguity Made Precise: A Comparative Foundation," *Journal of Economic Theory*, 102(2), 251–289.

GILBOA, I., AND D. SCHMEIDLER (1989): "Maxmin expected utility with non-unique prior," *Journal of Mathematical Economics*, 18(2), 141 – 153.

GOLLIER, C. (2001): "Wealth Inequality and Asset Pricing," *Review of Economic Studies*, 68(1), 181–203.

GORMAN, W. M. (1959): "Are Social Indifference Curves Convex?," *The Quarterly Journal of Economics*, 73(3), 485–496.

GRAAFF, J. (1957): *Theoretical Welfare Economics.* Cambridge University Press, Cambridge, UK.

KIHLSTROM, R., A. ROTH, AND D. SCHMEIDLER (1981): "Risk aversion and solutions to Nash's bargaining problem," in *Game theory and mathematical economics*, ed. by O. Moeschlin, and D. Pallaschke, pp. 65–71. North Holland.

KLIBANOFF, P., M. MARINACCI, AND S. MUKERJI (2005): "A Smooth Model of Decision Making under Ambiguity," *Econometrica*, 73(6), 1849–1892.

LERNER, A. P. (1934): "The Diagrammatical Representation of Demand Conditions in International Trade," *Economica*, 1(3), 319–334.

MACCHERONI, F., M. MARINACCI, AND A. RUSTICHINI (2006): "Ambiguity Aversion, Robustness, and the Variational Representation of Preferences," *Econometrica*, 74(6), 1447–1498.

MACHINA, M. J., AND D. SCHMEIDLER (1992): "A More Robust Definition of Subjective Probability," *Econometrica*, 60(4), 745–780.

MYERSON, R. B. (2004): *Probability Models for Economic Decisions*. Duxbury Press, Pacific Grove, CA.

RIGOTTI, L., AND C. SHANNON (2005): "Uncertainty and Risk in Financial Markets," *Econometrica*, 73(1), 203–243.

RIGOTTI, L., C. SHANNON, AND T. STRZALECKI (2008): "Subjective Beliefs and Ex-Ante Trade," *Econometrica*, 76(5), 1167–1190.

ROCKAFELLAR, R. T. (1970): *Convex analysis*. Princeton University Press, Princeton, N.J.

RUBINSTEIN, A., Z. SAFRA, AND W. THOMSON (1992): "On the Interpretation of the Nash Bargaining Solution and Its Extension to Non-Expected Utility Preferences," *Econometrica*, 60(5), 1171–1186.

SAMUELSON, P. A. (1956): "Social Indifference Curves," *Quarterly Journal of Economics*, 70(1), 1–22.

――― (1964): "Discussion," *American Economic Review*, 59(1), 88–96.

SCHMEIDLER, D. (1989): "Subjective Probability and Expected Utility without Additivity," *Econometrica*, 57(3), 571–587.

SEO, K. (2008): "Ambiguity and Second-Order Belief," Forthcoming, Econometrica.

SINISCALCHI, M. (2007): "Vector Expected Utility and Attitudes toward Variation," Forthcoming, *Econometrica*.

VICKREY, W. (1964): "Discussion," *American Economic Review*, 59(1), 88–96.

WILSON, R. (1968): "The Theory of Syndicates," *Econometrica*, 36(1), 119–132.

YAARI, M. E. (1969): "Some remarks on measures of risk aversion and on their uses," *Journal of Economic Theory*, 1(3), 315 – 329.