Preference identification

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Model

Alice (an experimenter)

Bob (a subject)

Model

- Alice presents Bob with choice problems:

“Hey Bob would you like x or y?”

x vs. y

- Bob chooses one alternative.
- Rinse and repeat $\rightarrow$ dataset of k choices.

Rationalization (roughly speaking)

A *rationalization* is a preference that would have generated the observed choices,

(Details later.)

Model

- An experimenter and a subject.
- Subject makes choices according to some $\succeq^*$, or utility u^*, on set X.
- Experimenter conducts a finite choice experiment of "size" k: k questions, each one a binary choice problem.
- Preference $\succeq_k$ or utility u_k as rationalizations or estimates.

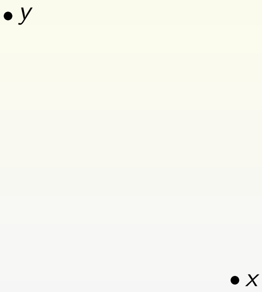
How are $\succeq_k$, $\succeq^*$, u_k and u^* related?

Example 1

Subject chooses among alternatives: $X = \mathrm{R}^n_+$.

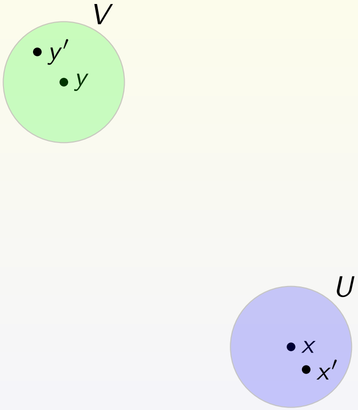
- Choices come from $\succeq^*$, a continuous preference.
- $\Sigma_i = \{x_i, y_i\}$.
- A *finite experiment:* choose an element from Σ_i, $i = 1, \ldots, k$.
- Assumption: $\Sigma_\infty = \cup_{k=1}^{\infty} \Sigma_k$ is dense.

Example 1



- $x \succ^* y$

Example 1



- $x \succ^* y$
- $U \succ^* V$
- $\exists x' \in U$ and $y' \in V$ s.t $\forall k$ $\exists$ rationalizing $\succeq_k$, with $y' \succ_k x'$
- But $x' \succ y'$. $\forall \succeq$ s.t. $\succeq$ is cont. and $\succeq|_{\Sigma_\infty} = \succeq^*|_{\Sigma_\infty}$.

Example 1: a "discontinuity."

- Infinite data ($\succeq^*$ on X): observe $\succeq^*$; so $x' \succ^* y'$
- "Limiting" infinite data ($\Sigma_\infty = \cup_{k=1}^{\infty} \Sigma_k$):
  $x' \succ y' \; \forall \succeq$ s.t. $\succeq |_{\Sigma_\infty} = \succeq^* |_{\Sigma_\infty}$.
- Finite data: $(\Sigma_1 \ldots, \Sigma_k)$
  can't rule out $y' \succ_k x'$, no matter how large k.

Lesson 1

No amount of finite data may correct a mistaken inference.

Even when the (limiting) infinite data set leaves no room for error.

Example 2

Let $X = \mathbb{R}^n_+$.

Fix a continuous preference $\succeq^*$ on X.

Proposition (informal)

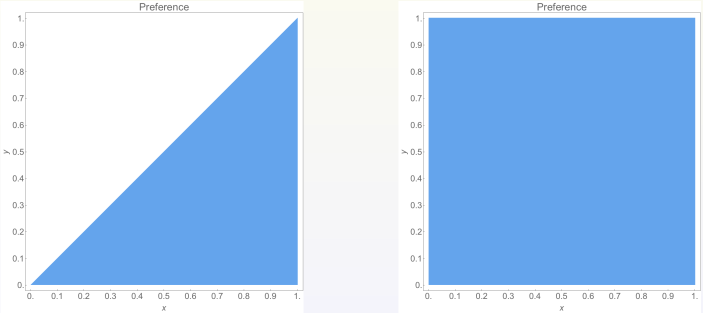
There exists locally non-satiated rationalizing $\succeq_k$ for each k s.t

$$\text{complete indifference } = \lim_{k \to \infty} \succeq_k$$

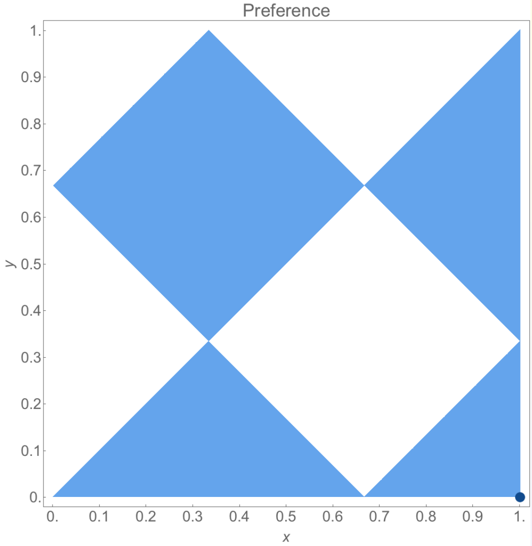
Example 2

Set of alternatives $X = [0, 1]$.

- Left: the subject prefers x to y iff $x \geq y$.
- Right: the subject is completely indifferent.



Example 2



Preference
1.
0.9
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0.6
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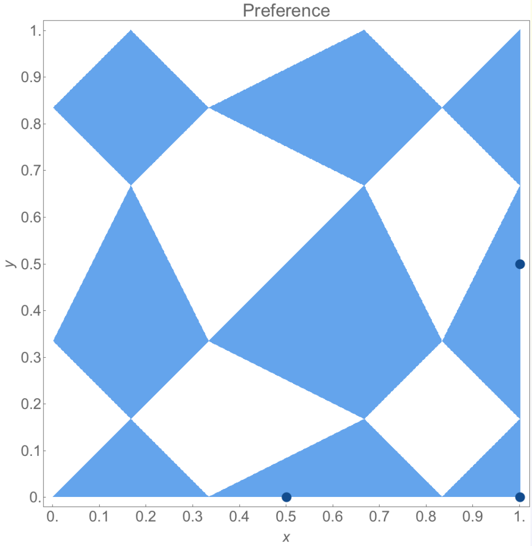
1.
x

n=1

Example 2



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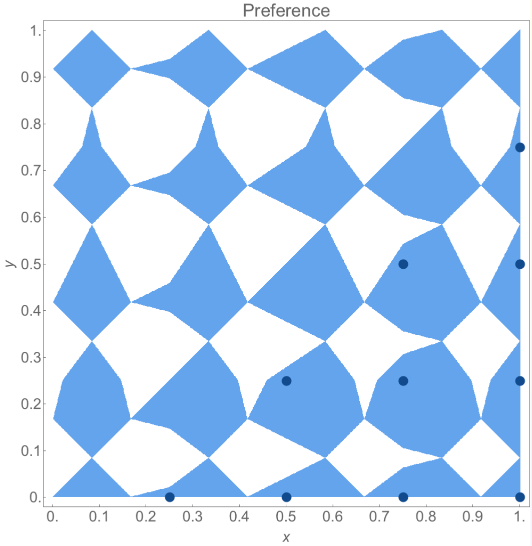
x

n=2

Example 2



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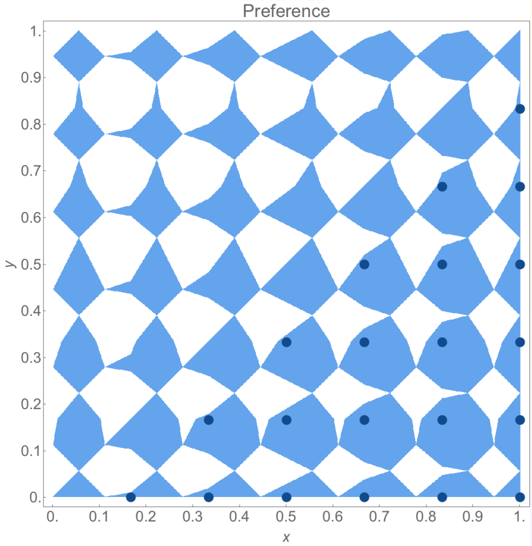
0.9
1.
x

n=4

Example 2



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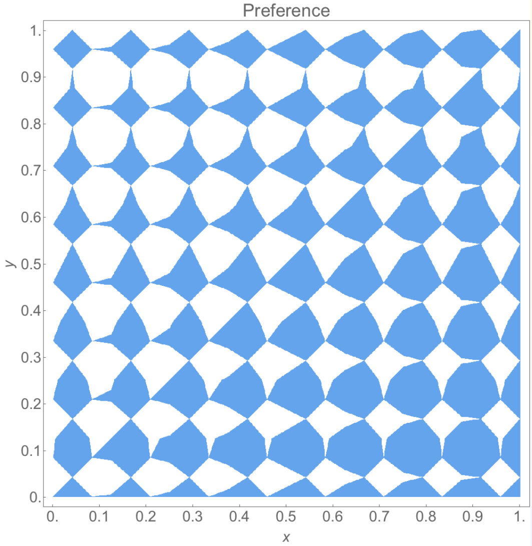
0.9
1.
x

n=6

Example 2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y
0.
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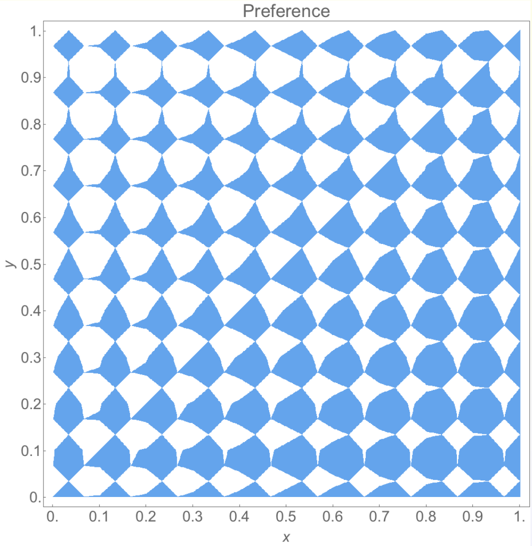
0.4
0.5
0.6
0.7
0.8
0.9
1.
x

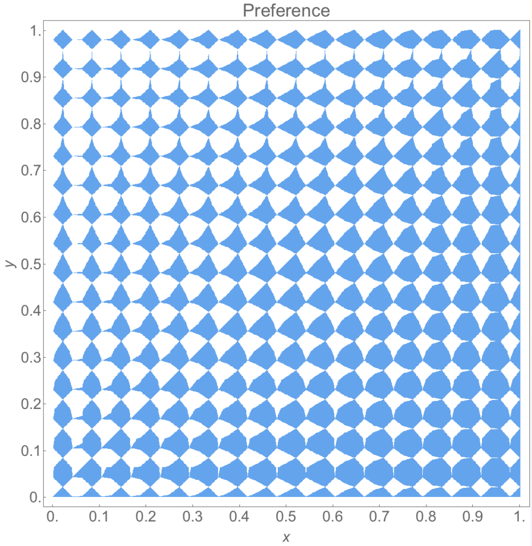
n=8

Example 2



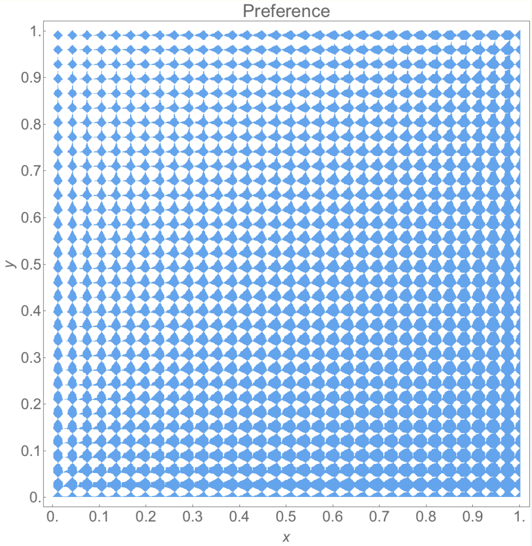
n=10

Example 2



n=16

Example 2



n=32

Lesson 2

Discipline matters.

Empiricism is dangerous.

Inevitable role for theory (a Cartesian imperative).

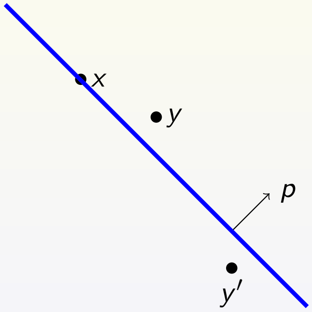
Example 3

Choice under uncertainty:

- State space $S = \{s_1, s_2\}$.
- Choice among monetary acts: $x \in \mathrm{R}^S$.
- Bob is risk-neutral subjective exp. utility maximizer.
- So $x \succeq^* y$ iff $p \cdot x \geq p \cdot y$.
- Preferences described by a prior $p \in \Delta(S)$.

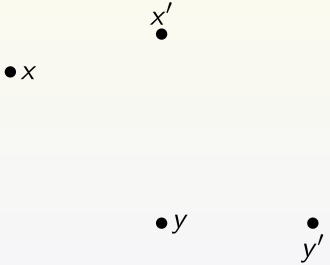
Example 3

Bob's preferences:



Example 3

Suppose y is chosen over x, and x' over y'.



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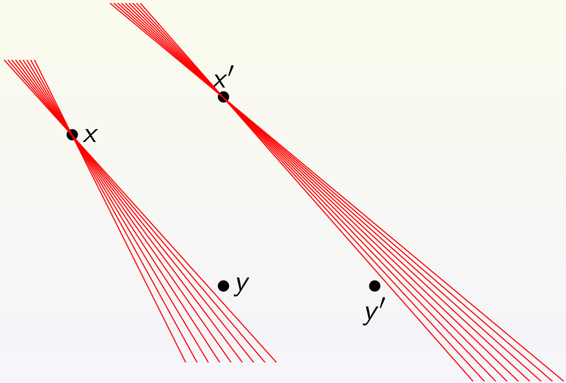

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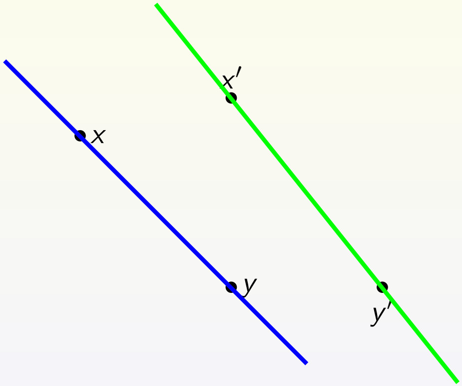

Bob's prior p must be steeper than the blue line, and flatter than the green.

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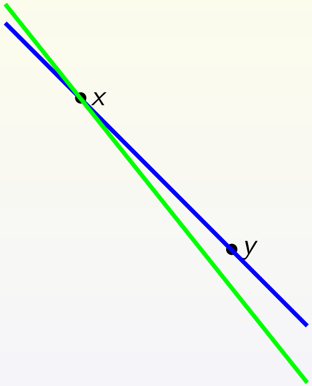

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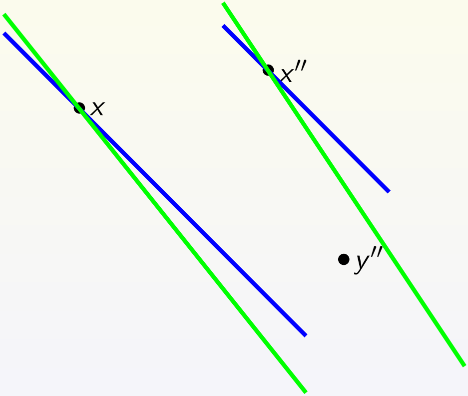


Narrows down unobserved comparison: $x'' \succ^* y''$.

Example 3

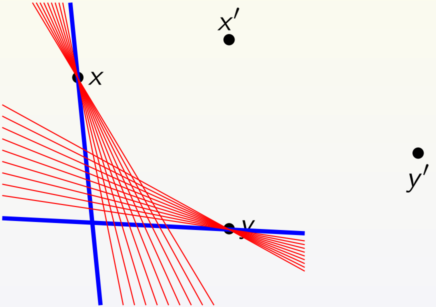
Suppose Alice instead uses the max-min model for Bob:

$$u(x) = \min\{p \cdot x : p \in \Pi\}$$

With two states, Π is described by four parameters. With more than two states, the model is non-parametric.
Then from $y \succ x$ she learns something about the slope of the worst-case priors.

Example 3

y is chosen over x, and x' over y'.



Example 3

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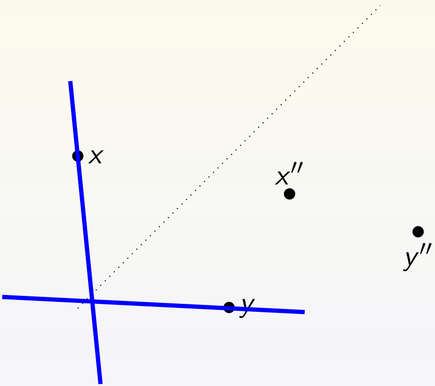


No inference for x'' and y''.

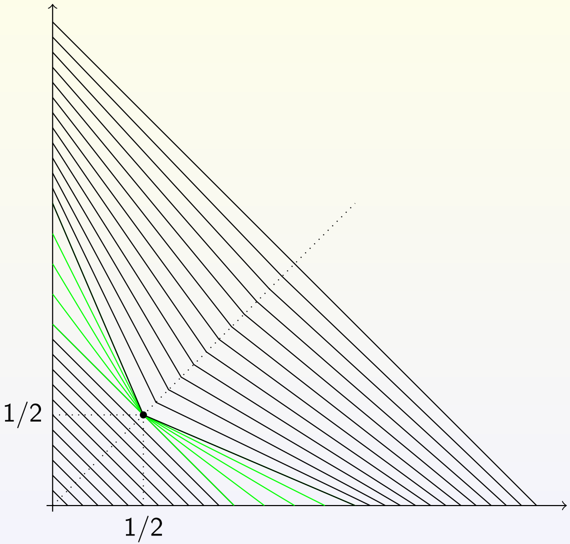
Lesson 3

A more flexible theory may lead to *overfitting.*

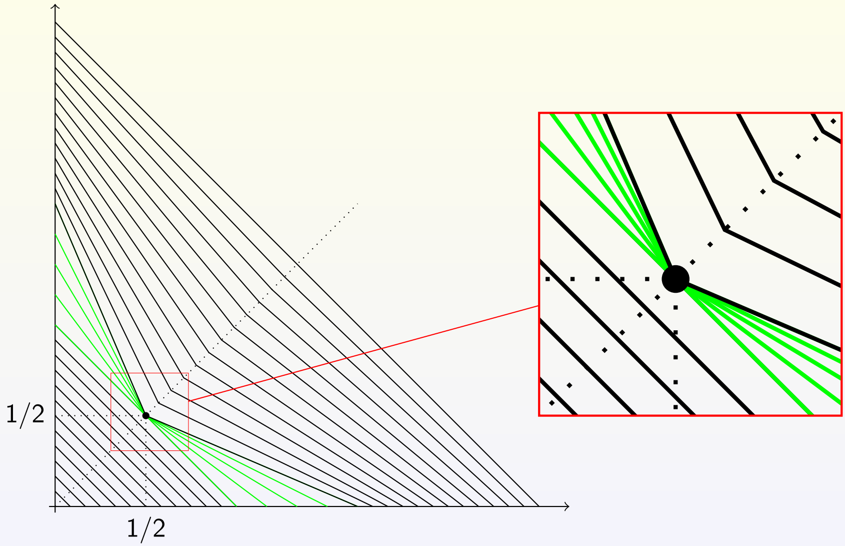
In fact max-min with $|S| \geq 3$ is "hopeless."

Any finite dataset will lead to poor out-of-sample predictions.

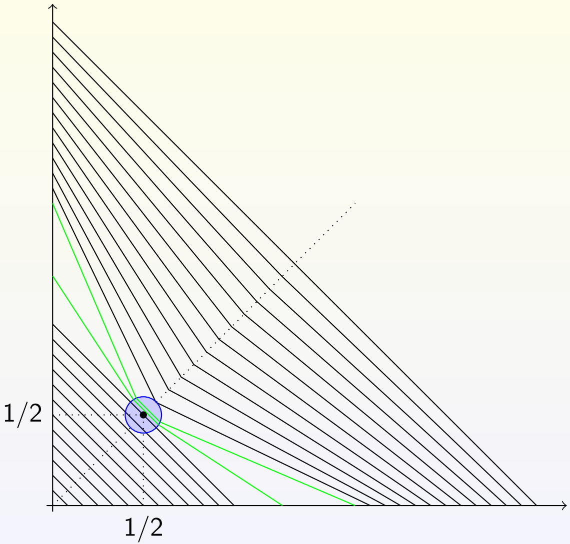
Example 4: Grodal's example



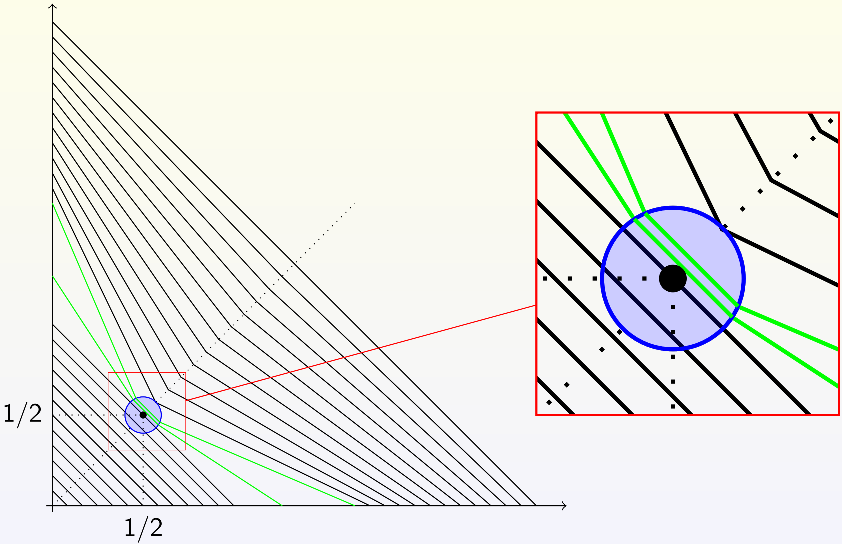
Example 4: Grodal's example



Example 4



Example 4



Model of preferences must be *closed*.

Can't allow for approximate behavior to "escape."

Example 5

- Let $X = [0,1]$, $\succeq^* = \geq$ and $u^*(x) = x$.
- For each k, let $\succeq_k = \geq$ and

$$u_k = \frac{x}{k}.$$

- Then $0 = \lim_k u_k$.
- But $\succeq_k = \succeq^*$ for all k!

Example 5

- Let $X = [0, 1]$, $\succeq^* = \geq$ and $u^*(x) = x$.
- For each k, let $\succeq_k = \geq$ and

$$u_k = \frac{x}{k}.$$

- Then $0 = \lim_k u_k$.
- But $\succeq_k = \succeq^*$ for all k!

(For $\varepsilon > 0$, can choose u_n with $\|u_n\|_\infty = 1$ or $\|u_n\|_1 = 1$ and $0 = \lim_n u_n(x)$ for all $x \in [0, 1 - \varepsilon]$.)

Lesson 5

Utility estimates are more delicate than preferences.

Must choose the right utility representation.

Lessons for DT

Typical result in decision theory:

"Utility representation iff axioms. Moreover, utility is unique."

Axioms ⇒ testable implications. But ignores overfitting problem.

Uniqueness ⇒ identification. But more is needed to ensure utility recovery from finite data.

Model

- Alternatives: A topological space X.
- Preference: A complete and continuous binary relation $\succeq$ over X
- $\mathcal{P}$ a set of preferences.

A pair $(X, \mathcal{P})$ is a preference environment.

Example: Expected utility preferences

- There are d prizes.
- X is the set of lotteries over the prizes, $\Delta^{d-1} \subset \mathrm{R}^d$.
- An EU preference $\succeq$ is defined by $v \in \mathrm{R}^d$ such that $p \succeq p'$ iff $v \cdot p \geq v \cdot p'$.
- $\mathcal{P}$ is set of all the EU preferences.

Experiment

Alice wants to recover Bob's preference from his choices.

- Binary choice problem : $\{x, y\} \subset X$.
- Bob is asked to choose x or y.
  Behavior encoded by a choice function $c(\{x, y\}) \in \{x, y\}$.
- If Bob's preference is $\succeq$ then $c(\{x, y\}) \succeq x$ and $c(\{x, y\}) \succeq y$.
- Partial observability: indifference is not observable.

Experiment

Alice gets finite dataset.

- Experiment of size k : $\Sigma^k = \{\Sigma_1, \ldots, \Sigma_k\}$ with $\Sigma_i = \{x_i, y_i\}$.
- Set of growing experiments: $\{\Sigma^k\} = \{\Sigma^1, \Sigma^2, \ldots\}$ with $\Sigma^k \subset \Sigma^{k+1}$.

Literature

Afriat's theorem and revealed preference tests: Afriat (1967); Diewert (1973); Varian (1982); Matzkin (1991); Chavas and Cox (1993); Brown and Matzkin (1996); Forges and Minelli (2009); Carvajal, Deb, Fenske, and Quah (2013); Reny (2015); Nishimura, Ok, and Quah (2017)

Recoverability: Varian (1982); Cherchye, De Rock, and Vermeulen (2011); Chambers, Echenique and Lambert (2021).

Consistency: Mas-Colell (1978); Forges and Minelli (2009); Kübler and Polemarchakis (2017); Polemarchakis, Selden, and Song (2017)

Identification: Matzkin (2006); Gorno (2019)

Econometric methods: Matzkin (2003); Blundell, Browning, and Crawford (2008); Blundell, Kristensen, and Matzkin (2010); Halevy, Persitz, and Zrill (2018)

OK, so far:

- $(X, \mathcal{P})$ preference env.
- c encodes choice
- Σ^k seq. of experiments

Rationalization

- A preference $\succeq$ weakly rationalizes the observed choices on Σ^k if $c(\{x,y\}) \succeq x$ and $c(\{x,y\}) \succeq y$ for all $\{x,y\} \in \Sigma^k$.
- A preference $\succeq$ strongly rationalizes the observed choices on Σ^k if $c(\{x,y\}) \succ z$ for $z \in \{x,y\}$, $z \neq c(\{x,y\})$, for all $\{x,y\} \in \Sigma^k$.

Topology on preferences

Choice of topology: closed convergence topology.

- Standard topology on preferences (Kannai, 1970; Mertens (1970); Hildenbrand, 1970).
- $\succeq_n \to \succeq$ when:
  1. For all $(x, y) \in \succeq$, there exists a seq. $(x_n, y_n) \in \succ_n$ that converges to (x, y).
  2. If a subsequence $(x_{n_k}, y_{n_k}) \in \succeq_{n_k}$ converges, the limit belongs to $\succeq$.
- If X is compact and metrizable, same as convergence under the Hausdorff metric.
- X Euclidean and $\mathcal{B}$ the strict parts of cont. weak orders. Then it's the smallest topology for which the set

$$\{(x, y, \succ) : x \in X, y \in X, \succ \in \mathcal{B} \text{ and } x \succ y\}$$

is open.

Topology on preferences

Lemma

Let X be a locally-compact Polish (separable and completely metrizable) space. Then the set of all continuous binary relations on X is a compact metrizable space.

Topology of compact convergence

Let $\{u_k\}$ be a sequence of functions,

$$u_k \colon X \to \mathrm{R}.$$

The sequence *convergences compactly* to $u \colon X \to \mathrm{R}$ if for every compact $K \subseteq X$,

$$u_k|_K \to u|_K$$

uniformly.

Turn out to be the right topology for utility functions when preferences are endowed with the closed convergence topology (the reason being that $u_k \to u$ and $x_k \to x$ then $u_k(x_k) \to u(x)$).

Results

Let X be

- $X = \mathbb{R}^n$.
- or $X = \Delta([a, b])^\Omega$ (set of "monetary" Anscombe-Aumann acts) with finite Ω.

Obs.

- Objective monotonicity.
- Connection between order and topology on X.
- Some of our results are more general.

Results

A sequence of experiments $\{\Sigma^k\}$, with $\Sigma^k = \{\Sigma_1, \ldots, \Sigma_k\}$, is exhaustive when:

1. $\bigcup_{i=1}^{\infty} \Sigma_i$ is dense in X.
2. For all $x, y \in \bigcup_{i=1}^{\infty} \Sigma_i$ with $x \neq y$, there exists i s.t $\Sigma_i = \{x, y\}$.

Results

Theorem

Let

- $\succeq^*$ be monotone and cont.;
- $\succeq_k$ strongly rationalize the k-sized choice data generated by $\succeq^*$.

Then,

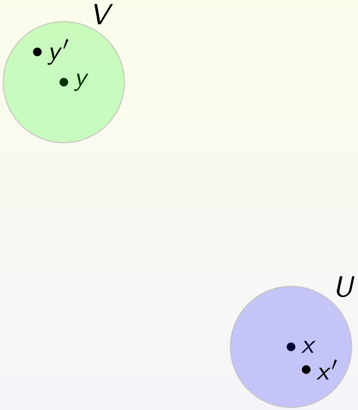
- $\succeq_k \to \succeq^*$ (in the topology of closed convergence).
- For any utility u^* for $\succeq^*$ $\exists$ u_k for $\succeq_k$ s.t $u_k \to u^*$ (in the topology of compact convergence).

Discussion.

- Monotonicity.
- Convergence of *any arbitrary* preference rationalization.
- Utility *can't be arbitrary*. Only get convergence of selected utility estimates. Require an identification theorem for each specific theory.

Why does monotonicity help?

Recall Example 1



- $x \succ^* y$
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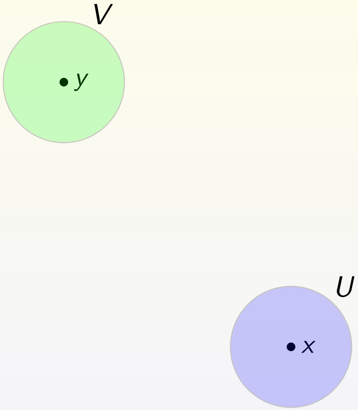
- But $x' \succ y'$. $\forall \succeq$ s.t. $\succeq$ is cont. and $\succeq|_B = \succeq^*|_B$.

Monotone rationalizations.



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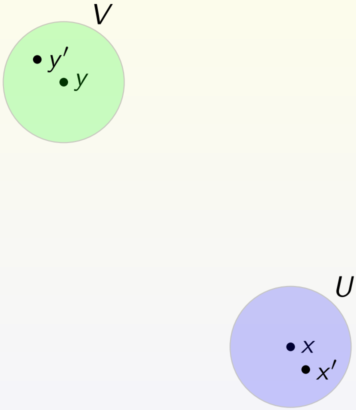
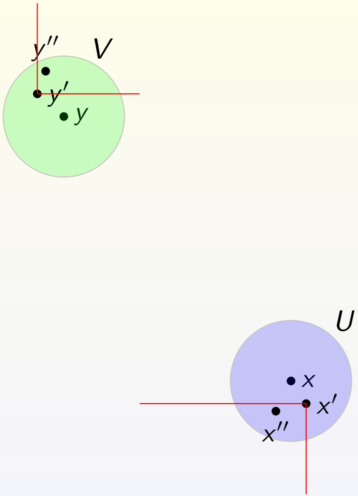


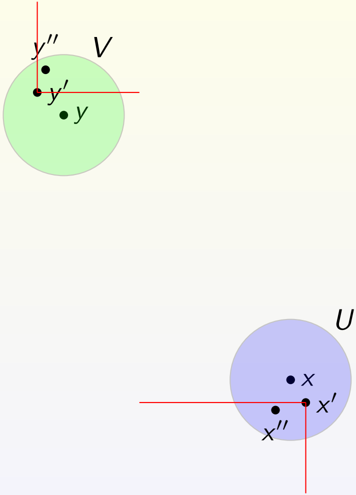
- $x \succ^* y$
- $U \succ^* V$
- Let $(x', y') \in U \times V$.

Monotone rationalizations.



- $x \succ^* y$
- $U \succ^* V$
- Let $(x', y') \in U \times V$.
- $\implies \exists x'', y'' \in B$
- $x'' \leq x'$
- $y' \leq y''$

Monotone rationalizations.



- $x \succ^* y$
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- Let $(x', y') \in U \times V$.
- $\implies \exists x'', y'' \in B$
- $x'' \leq x'$
- $y' \leq y''$
  $\implies x' \geq x'' \succ_k y'' \geq y'$

Weak rationalizations

A preference $\succeq$ is *locally strict* if

$x \succeq y \implies$ in every nbd. of (x, y), there exists (x', y') with $x' \succ y'$

(Border and Segal, 1994).

Weak rationalizations

Let $X \subseteq \mathrm{R}^n$. and $\mathcal{P}$ be a closed set of locally strict preferences on X.

Theorem

Let $\succeq_k \in \mathcal{P}$ weakly rationalize the k-sized choice data.

- Then there is a preference $\succeq^* \in \mathcal{P}$ s.t $\succeq_k \to \succeq^*$.
- The limiting preference is unique: if, for every k, $\succeq'_k \in P$ rationalizes the k-data, then the same limit $\succeq'_k \to \succeq^*$ obtains.

Obs. that $\succeq^*$ generating the choice is not a hypothesis. May view this result as a definition of preference.
(This result is in Chambers-Echenique-Lambert (2021))

Utility functions

Utility representations

We need a canonical utility representation.

Here we use the "equal coordinates" idea: a set M on which all preferences agree.

For $X = \mathrm{R}^n$ M, is the ray of equal coordinates.
For $X = \Delta([a, b])$, M is $[a, b]$.

For the talk, assume $X = \mathrm{R}^n$.

Model

Let

- $\mathcal{U}$ be the set of st. monotone and cont. utility functions on X.
- $\mathcal{R}^{\text{mon}}$ be the set of preferences which are st. monotone and cont.

Homeomorphism

Let $\Phi : \mathcal{U} \rightarrow \mathcal{R}^{\text{mon}}$ such that $\Phi(u)$ is the preference represented by $u \in \mathcal{U}$.

Equivalence relation $\simeq$ on $\mathcal{U}$;

$\hat{\Phi} : \mathcal{U}/ \simeq \rightarrow \mathcal{R}$ is defined in the natural way.

Theorem

$\hat{\Phi}$ is a homeomorphism.

Homeomorphism

Homeomorphism tells us how to go from recovered preferences to utilities, and from recovered utilities to preferences. . .

Recall:

Let $X \subseteq \mathrm{R}^n$. and $\mathcal{P}$ be a closed set of locally strict preferences on X.

Theorem

Let $\succeq_k \in \mathcal{P}$ weakly rationalize the k-sized choice data.

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Ideas behind the thm

Lemma

The set of all continuous binary relations on X is a compact metrizable space.

Lemma

If $A \subseteq X \times X$, then $\{\succeq \in X \times X : A \subseteq \succeq\}$ is closed.

Identification

Lemma

Consider an exhaustive set of experiments with binary choice problems $\{x_k, y_k\}$, $k \in \mathbf{N}$. Let $\succeq$ be any complete binary relation, and $\succeq_A$ and $\succeq_B$ be locally strict preferences. If, for all k, $x_k \succeq_A y_k$ and $x_k \succeq_B y_k$ whenever $x_k \succeq y_k$, then $\succeq_A = \succeq_B$.

Statistical model

Given $(X, \mathcal{P})$. We change:

- How subjects make choices: they do not exactly follow a preference, but randomly deviate from it.
- How experiments are generated.

Statistical model

1. In a choice problem, alternatives drawn iid according to sampling distribution λ.
2. Subjects make "mistakes."
   Upon deciding on $\{x, y\}$, a subject with preference $\succeq$ chooses x over y with probability $q(\succeq; x, y)$ (error probability function).
3. Only assumption: if $x \succ y$ then $q(\succeq; x, y) > 1/2$.
4. "Spatial" dependence of q on x and y is arbitrary.

Estimator

Kemeny-minimizing estimator: find a preference in $\mathcal{P}$ that minimizes the number of observations inconsistent with the preference.

- "Model free:" to compute estimator don't need to assume a specific q or λ.
- May be computationally challenging (depending on $\mathcal{P}$).

To sum up:

ASSUMPTION 1 : X is a locally compact, separable, and completely metrizable space.

ASSUMPTION 2 : $\mathcal{P}$ is a closed set of locally strict preferences.

ASSUMPTION 3' : λ has full support and for all $\succeq \in \mathcal{P}$, $\{(x, y) : x \sim y\}$ has λ-probability 0.

Second main result

Theorem

Under Assumptions (1), (2), (3'), if the subject's preference is $\succeq^* \in \mathcal{P}$ and $\succeq_n$ is the Kemeny-minimizing estimator for Σ_n, then, $\succeq_n \to \succeq^*$ in probability.

Finite data

- Our paper is about finite data.
- Finite data but large samples
- How large?

Convergence rates: Digression

The VC dimension of $\mathcal{P}$ is the largest cardinality of an experiment that can always be rationalized by $\mathcal{P}$.

A measure of how flexible $\mathcal{P}$; how prone it is to overfitting.

Convergence rates: Digression

- Think of a game between Alicia and Roberto
- Alicia defends $\mathcal{P}$; Roberto questions it.
- Given is k
- Alicia proposes a choice experiment of size k
- Roberto fills in choices adversarily.
- Alicia wins if she can rationalize the choices using $\mathcal{P}$.
- The VC dimension of $\mathcal{P}$ is the largest k for which Alicia always wins.

Convergence rates

- Let ρ be a metric on preferences.

Theorem 2 (Part B)

Under the same conditions as in Part A,

$$N(\eta, \delta) \leq \frac{2}{r(\eta)^2} \left(\sqrt{2/\delta} + C\sqrt{\mathbf{VC}(\mathcal{P})} \right)^2$$

with C a universal constant.

Convergence rates

- Let ρ be a metric on preferences.
- $N(\eta, \delta)$: smallest value of N such that for all $n \geq N$, and all subject preferences $\succeq^* \in \mathcal{P}$,

$$\Pr(\rho(\succeq_n, \succeq^*) < \eta) \geq 1 - \delta.$$

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- $\mu(\succeq'; \succeq)$: probability that the choice of a subject with preference $\succeq$ is consistent with preference $\succeq'$.

$$r(\eta) = \inf \left\{ \mu(\succeq; \succeq) - \mu(\succeq'; \succeq) : \succeq, \succeq' \in \mathcal{P}, \rho(\succeq, \succeq') \geq \eta \right\}.$$

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with C a universal constant.

Expected utility

1. X is the set of lotteries over d prizes.
2. $\mathcal{P}$ is the set of nonconstant EU preferences: there are always lotteries p, p' such as p is strictly preferred to p'.

This preference environment satisfies Assumptions 1 and 2.

Suppose: there is $C > 0$ and $k > 0$ s.t

$$q(x, y; \succeq) \geq \frac{1}{2} + C(v \cdot x - v \cdot y)^k,$$

when $x \succeq y$ and v represents $\succeq$.

Expected utility

Under these assumptions, we can bound $r(\eta)$ and $\mathbf{VC}(\mathcal{P})$, which implies

$$N(\eta, \delta) = O\left(\frac{1}{\delta \eta^{4d-2}}\right).$$

Other examples: Cobb-Douglas, CES, and CARA subjective EU preferences, and intertemporal choice with discounted, Lipschitz-bounded utilities.

Monotone preferences

- K be a compact set in $X \equiv \mathbb{R}^d_{++}$, and fix $\theta > 0$.
- $\mathcal{P}$ has finite VC-dimension and is identified on K
- λ is the uniform probability measure on $K^{\theta/2}$,
- q satisfies: probability of choosing y instead of x when $x \succ y$ is a function of $\|x - y\|$,

Proposition

The Kemeny-minimizing estimator is consistent and, as $\eta \to 0$ and $\delta \to 0$,

$$N(\eta, \delta) = O\left(\frac{1}{\eta^{2d+2}} \ln \frac{1}{\delta}\right).$$