

# Closure Operators: Complexity and Applications to Classification and Decision-making.

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July 12, 2022

Let  $X$  be a finite set.

A function  $f : 2^X \rightarrow 2^X$  is a *closure operator* if it satisfies

1. **Extensivity:**  $A \subseteq f(A)$  and  $f(\emptyset) = \emptyset$ .
2. **Idempotence:**  $f(f(A)) = f(A)$ .
3. **Monotonicity:**  $A \subseteq B$  implies  $f(A) \subseteq f(B)$ .

Two settings modeled by closure operators:

1. Machine Learning: Classification.
2. Decision theory: Choice over menus.

We offer a discussion of complexity of closure operators, with applications in ML and DT.

# Classification

Consider the data:

$$X = \{\text{Homer, Marge, Bart, Lisa, Maggie, SLH, Snowb.}\}$$

One classifier recognizes the humans:  $f(A) = \{H, M, B, L, g\}$  if  $A \neq \emptyset$  and  $A \subseteq \{H, M, B, L, g\}$  but  $f(A) = X$  if  $A \ni S, n$ .

A second classifier recognizes adults:  $f(H) = f(M) = f(H, M) = \{H, M\}$ , while  $f(H, B) = X$  and  $f(L, g) = X$  (for example).

A third classifier would recognize males and children, so for example  $f(\{B\}) = \{B\}$  while  $f(\{B, S\}) = \{H, B, S\}$ .

## Choice over menus (Kreps '79)

For Homer we may have

$$\begin{aligned} \{\text{Beer, G\&T, Lemonade}\} &\sim \{\text{Beer, Lemonade}\} \sim \{\text{Beer, G\&T}\} \\ &\succ \{\text{G\&T}\} \sim \{\text{Lemonade}\} \end{aligned}$$

And for Marge:

$$\{\text{Beer, G\&T, Lemonade}\} \succ \{\text{Beer, Lemonade}\} \sim \{\text{Beer, G\&T}\}$$

## Choice over menus (Kreps '79)

This is modeled through closure operators.

For Homer:

$$f(A) = \{\text{Beer, G\&T, Lemonade}\}$$

if  $A$  contains beer.

For Marge:

$$f(A) \supseteq \{\text{G\&T, Lemonade}\}$$

only if

$$A \supseteq \{\text{G\&T, Lemonade}\}$$

# Basic definitions

A binary relation on  $X$  is:

A *weak order* on  $X$  if it is complete and transitive (meaning that it orders all elements of  $X$ );

A *partial order* if it is reflexive, transitive and anti-symmetric ( $\forall x, y \in X$  if  $x \preceq y$  and  $y \preceq x$  then  $x = y$ );

A *total order* if it is a complete partial order.

# Definitions from convex geometry

Denote the set of all weak orders on  $X$  by  $\mathcal{R}$ .

The *support function* of  $A \subseteq X$  is the function  $h_A : \mathcal{R} \rightarrow 2^X$  defined by

$$h_A(\succsim) = \{x \in A \mid x \succsim y \ \forall y \in A\}.$$

The *support half-space* of  $A \neq \emptyset$  with respect to  $\succsim \in \mathcal{R}$  by

$$H(\succsim, h_A) = \{x \in X \mid h_A(\succsim) \succsim x\}.$$

(Observe the analogy w/convex analysis. Here the set  $\mathcal{R}$  serves the same role as the dual of  $X$ .)



## Definition

A set  $A \subseteq X$  is *closed* with respect to a closure operator  $f : 2^X \rightarrow 2^X$ , if  $A = f(A)$ .

The set  $f(2^X)$  of all closed sets with respect to the closure operator  $f$  is the *topology* defined by  $f$ , and is denoted by  $S(f)$ .

The terminology of closed sets and topology is justified by the following well-known result.

## Lemma

Let  $f : 2^X \rightarrow 2^X$  be a closure operator on  $X$ . The set of closed sets  $S(f)$  is closed under intersection and contains  $\emptyset$  and  $X$ . Indeed,  $S(f)$  endowed with the meet and join operators  $A \wedge B = A \cap B$  and  $A \vee B = f(A \cup B)$  is a lattice that has  $\emptyset$  and  $X$  as its (respectively) smallest and largest elements.

Conversely, if  $S$  is any subset of  $2^X$  that is closed under intersections and contains  $\emptyset$  and  $X$ , then there is a unique closure operator  $f_S : 2^X \rightarrow 2^X$  such that  $S(f_S) = S$ .

# Example of closure operators

1. The *identity operator* is the closure operator  $I : 2^X \rightarrow 2^X$  such that  $I(A) = A$  for every  $A \in 2^X$ .
2. The *trivial closure operator* is defined as the closure operator  $f : 2^X \rightarrow 2^X$  such that  $f(A) = X$  for every nonempty  $A \in 2^X$ .

# Example of closure operators

3. A *binary classifier* is an operator  $f_C$ , associated to a set  $C \subseteq X$ .

$$f_C(A) = \begin{cases} \emptyset & \text{if } A = \emptyset \\ C & \text{if } \emptyset \neq A \subseteq C \\ X & \text{otherwise} \end{cases}$$

4. Given a function  $u : X \rightarrow \mathbf{R}$ ,

$$f_u(A) = \{x \in X : u(x) \leq \max\{u(x') : x' \in A\}\}$$

is the *strategically rational* operator defined from the function  $u$ .

# Example of closure operators

First note that a binary classifier gives rise to closed sets  $S(f_C) = \{\emptyset, C, X\}$ , which are the smallest kind of non-trivial topology possible.

We shall think of these as simple classifiers.

Binary classifiers are a special case of strategically rational operators: Indeed for a given  $C \subseteq X$  we may define  $u = \mathbf{1}_{X \setminus C}$ , and observe that  $f_C = f_u$ .

A key motivation is the next result, due to Chambers, Miller, and Yenmez (2020).

## Theorem

A function  $f : 2^X \rightarrow 2^X$  is a closure operator iff there exist weak orders  $\succsim_1, \dots, \succsim_k$  on  $X$ , such that

$$f(A) = \bigcap_{i \in \{1, \dots, k\}} H(\succsim_i, h_A). \quad (1)$$

(See also Richter-Rubinstein (2015) and Edelman and Jamison (1985))

We may rephrase this result as follows:

## Corollary

A function  $f : 2^X \rightarrow 2^X$  is a closure operator iff there exist strategically rational operators  $g_{u_i}$ ,  $1 \leq i \leq k$ , so that

$$f(A) = \bigcap_{i \in \{1, \dots, k\}} g_{u_i}(A). \quad (2)$$

Our first result elaborates on the above:

## Theorem

Let  $g_1, \dots, g_k$  be closure operators on  $X$ , then  $f : 2^X \rightarrow 2^X$  defined by

$$f(A) = \bigcap_{i \in \{1, \dots, k\}} g_i(A)$$

is a closure operator. We say that  $f$  is *generated* from  $g_1, \dots, g_k$ .

Moreover, if  $f, g_1, \dots, g_k$  are closure operators on  $X$  then  $f$  is generated from  $g_1, \dots, g_k$  iff

1.  $S(g_i) \subseteq S(f)$  for all  $i \in \{1, \dots, k\}$ ;
2. and if  $A \in S(f)$  and  $x \notin A$ , then there exists a closure operator  $g_i \in \{g_1, \dots, g_k\}$  such that  $x \notin g_i(A)$ .



If  $f$  is generated by  $g_1, \dots, g_k$ , then topology  $S(f)$  is finer than that of any constituent operator  $S(g_i)$ .

The second condition is similar to the separation property in linear spaces: If  $A$  is a closed set of  $f$  and  $x \notin A$ , then there should be a separating classifier  $g_i \in \{g_1, \dots, g_k\}$  that detects that  $x$  is not in the closure of  $A$  with respect to  $g_i$ .

## Definition

The operator  $f$  is *more complex* than the operator  $g$  if  $S(g) \subseteq S(f)$ .

So that a more complex operator induces a finer topology. The “more complex than” relation is, we think natural, but it induces an incomplete partial order on operators, and will render some pairs of operators incomparable.

We consider two ways of completing the complexity binary relations.

- ▶ The *minimum number of weak orders* (MNWO) of an operator  $f$  is the smallest number  $n$  so there exists weak orders  $\succeq_1, \dots, \succeq_n$  with  $f = \bigcap_{i=1}^n g_{\succeq_i}$ .
- ▶ The *minimum number of binary classifiers* (MNBC) of an operator  $f$  is the smallest number  $n$  so there exists subsets  $C_1, \dots, C_n$  of  $X$  with  $f = \bigcap_{i=1}^n g_{C_i}$ .

We may think of strategically rational operators as simple because they do not exhibit a preference for flexibility. Or because they reflect a one-dimensional property (like size, color, being closer to the entrance of a supermarket). This motivates the use of MNWO.

Think of MNBC as length of the binary code needed to describe the topology of the closure operator.

Let  $P(f)$  be the elements of the topology  $S(f)$  that are not the intersection of other closed sets in  $S(f)$ .

Let  $B(f)$  be the elements in  $P(f)$  other than  $\emptyset$  and  $X$ .

## Theorem

Let  $f$  be a closure operator, then the MNWO of  $f$  is equal to the width of  $P(f)$ , and the MNBC is equal to the cardinality of  $B(f)$ .

Suppose that  $\succsim$ , a preference relation over  $2^X$ , captures choices made over menus. The indifference relation derived from  $\succsim$  is denoted by  $\sim$ , so that  $A \sim B$  when  $A \succsim B$  and  $B \succsim A$ .

We entertain the following axioms on  $\succsim$ :

1. Desire for flexibility:  $B \subseteq A$  implies  $A \succsim B$ ,
2. Ordinal submodularity:  $A \sim A \cup B$  implies that for all  $C$ ,  
 $A \cup C \sim A \cup B \cup C$ .

These were introduced by Kreps (1979)

There are two possible approaches.

The first is to define the function  $f : 2^X \rightarrow 2^X$  from preference  $\succsim$  by

$$f(A) = \bigcup_{B \in 2^X, A \sim A \cup B} B. \quad (3)$$



The second approach is proposed by Dekel-Lipman-Rustichini in a model of choice over lotteries, using the resulting linear structure.

The key property in this approach is that a closure operator is given by a convex hull operator, so that  $A \sim f(A)$  for all menus  $A$ .

### Definition

$\succsim$  *respects*  $f$  if  $A \sim f(A)$  for every menu  $A \in 2^X$ .

Following the first approach we may show (under the two axioms):

1.  $f$  is a closure operator,
2.  $\succsim$  respects  $f$ ,
3.  $A \sim A \cup B$  if and only if  $f(B) \subseteq f(A)$ ,
4.  $f(B) \subset f(A)$ , then  $A \succ' B$ .

We obtain the following version of Kreps' result:

## Theorem

Let  $\succsim$  be a preference relation over  $2^X$  that satisfies desire for flexibility and ordinal submodularity, and let  $f$  be defined using Equation (3). Then  $\exists U : X \times S(f) \rightarrow \mathbb{R}$ , and a st. inc.  $u : \mathbb{R}^S \rightarrow \mathbb{R}$  s.t

$$u([\max_{a \in A} U(a, s)]_{s \in S}) \quad (4)$$

represents  $\succsim$ . The minimum number of states (cardinality of  $S(f)$ ) needed for the representation is precisely the MNWO of the associated operator  $f$ , which is  $|P(f)|$ .

If we instead adopt a given closure operator  $f$ , we obtain an analogue to the result in DLR:

## Theorem

Suppose a closure operator  $f$ , and a preference  $\succsim$  over  $2^X$  that respects  $f$ . Then  $\exists$  state space  $S$ , where  $S = S^+ \cup S^-$  with  $S^+ \cap S^- = \emptyset$  and has cardinality at most  $2(|S(f)| - 1)$ , and a state-dependent utility  $U : X \times S \rightarrow \mathbb{R}$  s.t

$$U(A) = \sum_{s \in S^+} \max_{a \in A} U(a, s) - \sum_{s \in S^-} \max_{a \in A} U(a, s) \quad (5)$$

represents  $\succsim$ .

## Corollary

For every preference ordering  $\succsim$  over the set of menus, there exists a representation as in Equation 5 with at most  $2 \times (2^{|X|} - 1)$  states.