Matching Markets

- Introduction: One-to-one matchings
- A Solution to Matching with Preferences over Colleagues
  Federico Echenique (Caltech) and Mehmet Yenmez (Stanford)

The Model

- a finite set W of workers,
- a finite set F, disjoint from W, of firms,
- a preference profile $P = (P(a))_{a \in W \cup F}$, where $P(a)$ is a strict preference relation over $F \cup \{\emptyset\}$ if $a \in W$, and over $W \cup \{\emptyset\}$ if $a \in F$.

Notation: $b' R(a) b$ if $b' = b$ or $b' P(a) b$.

A *matching* μ is a mapping from $F \cup W$ into $F \cup W \cup \{\emptyset\}$ s.t.

1. $\mu(w) \in F \cup \{\emptyset\}$.
2. $\mu(f) \in W \cup \{\emptyset\}$.
3. $f = \mu(w)$ iff $w = \mu(f)$.

Stability

- A matching μ is *individually rational* if $\mu(a)R(a)\emptyset \; \forall a$.
- A pair (w, f) *blocks* μ if $w \neq \mu(f)$,

$$wP(f)\mu(f) \text{ and } fP(w)\mu(w).$$

- A matching μ is *stable* if it is individually rational and there is no pair that blocks μ.

Set of stable matchings = the core

A *prematching* ν is a mapping from $F \cup W$ into $F \cup W \cup \{\emptyset\}$ s.t.

1. $\nu(w) \in F \cup \{\emptyset\}$.
2. $\nu(f) \in W \cup \{\emptyset\}$.

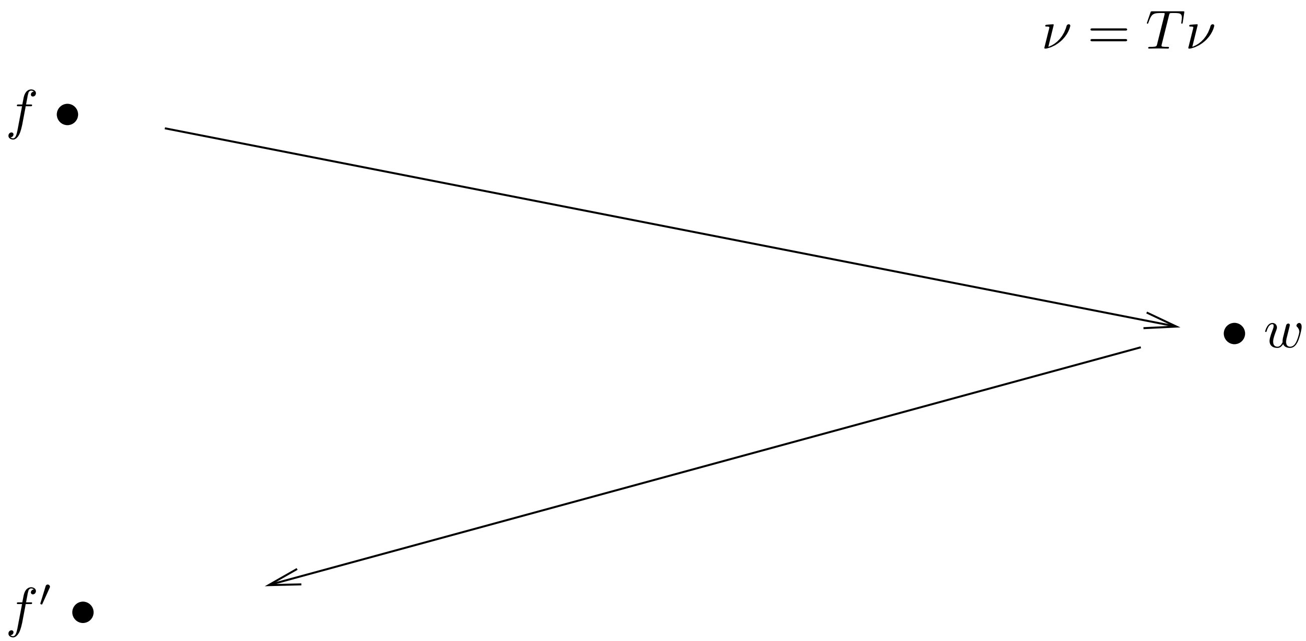
Let $\mathcal{V}$ = set of all prematchings.

A prematching is a fantasy

Construct $T : \mathcal{V} \to \mathcal{V}$.

- $U(f, \nu) = \{w : fR(w)\nu(w)\} \cup \{\emptyset\}$
- $V(w, \nu) = \{f : wR(f)\nu(f)\} \cup \{\emptyset\}$

$$\begin{aligned} (T\nu)(f) &= \max_{P(f)} U(f, \nu) \\ (T\nu)(w) &= \max_{P(w)} V(w, \nu) \end{aligned}$$


$\nu = T\nu$
f
w
f'

Order prematchings by $\leq_F$:

$\nu \leq_F \nu'$ iff

- $\nu'(f) R(f) \nu(f)$ for all f
- $\nu(w) R(w) \nu'(w)$ for all w

Let $\nu \leq_F \nu'$

$$\begin{aligned} w \in U(f,\nu) &\Rightarrow fR(w)\nu(w)R(w)\nu'(w) \\ &\Rightarrow w \in U(f,\nu') \end{aligned}$$

So $U(f,\nu) \subseteq U(f,\nu')$.

Similarly, $V(w,\nu') \subseteq V(w,\nu)$.

T is monotone increasing.

$$\mathcal{E} = \{\nu : \nu = T\nu\}$$

- $\mathcal{E}$ is a nonempty lattice
- T-algorithm finds a matching in $\mathcal{E}$.

Matching with Preferences over Colleagues

The Model. $\langle C, S, P \rangle$

- a set C of colleges
- a set S of students
- preferences $P(c)$ over 2^S, for each c
  preferences $P(s)$ over $\left(C \times 2^S\right) \cup \{(\emptyset, \emptyset)\}$

$$S_s = \{S' \subseteq S : S' \ni s\}$$

A *matching* μ is a mapping on $C \cup S$ s.t.

- $\mu(s) \in C \times S_s \cup \{(\emptyset, \emptyset)\}$
- $\mu(c) \in 2^S$
- $s \in \mu(c) \;\Rightarrow\; \mu(s) = (c, \mu(c))$
- $\mu(s) = (c, S') \;\Rightarrow\; \mu(c) = S'$.

$(B, c) \in 2^S \times C$ *blocks** μ if

$B \cap \mu(c) = \emptyset$

$\exists A \subseteq \mu(c)$ s.t. $\forall s' \in A \cup B, (c, A \cup B)P(s')\mu(s')$

$A \cup BP(c)\mu(c)$.

μ is in the *core* if it is IR and there is no block* of μ.

Example – empty core.

$C = \{c_1, c_2\}$, $S = \{s_1, s_2, s_3\}$

$$P(c_1): \quad s_1 s_2, s_1 s_3, s_1, s_2$$

$$P(c_2): \quad s_2 s_3, s_3, s_2$$

$$P(s_1): \quad (c_1, s_1 s_2), (c_1, s_1 s_3), (c_1, s_1)$$

$$P(s_2): \quad (c_2, s_2 s_3), (c_1, s_1 s_2), (c_1, s_2), (c_2, s_2)$$

$$P(s_3): \quad (c_1, s_1 s_3), (c_2, s_2 s_3), (c_2, s_3)$$

Need very strong assumptions to guarantee nonemptyness.

RESULTS:

- Algorithm finds the core match., if the exist.
- Algorithm is efficient when we can ensure nonemptyness.
- “Partial” solutions.

Fixed-point approach.

- prematchings
- T
- fixed points of T = core

$$
\begin{aligned}
U(c,\nu) &= \{S' \subseteq S : \forall s \in S', (c,S')R(s)\nu(s)\} \\
V(s,\nu) &= \big\{(c,S') \in C \times 2^S : s \in S', \forall s' \in S' \backslash \{s\} (c,S')R(s')\nu(s') \\
&\quad \text{and} \;\; S'R(c)\nu(c)\} \cup \{\emptyset \times \emptyset\}
\end{aligned}
$$

$(T\nu)(a) = \max_{P(a)} \ldots$

Theorem. *The core is the set of fixed points of T.*

- $\nu \leq \nu'$ if everyone prefers ν'
- T is decreasing
- T^2 is increasing

$\mathcal{E}(T^2)$ *is a nonempty complete lattice*

Algorithm: find matchings in $\mathcal{E}(T^2)$.

Will find the core, if nonempty.

May miss some matchings in $\mathcal{E}(T^2)\backslash\mathcal{E}(T)$.

Partial solutions

μ is in the *core with singles* if, for any block* (c, D) of μ,

$$\mu(c) = \emptyset$$
$$\forall s \in D \ \mu(s) = (\emptyset, \emptyset)$$

Let μ be a matching.

Theorem. $\mu \in \mathcal{E}(T^2) \Rightarrow \mu$ *is in the core with singles.*

Let μ be a matching w/no single agents.

Corollary. μ *is a core matching iff* $\mu \in \mathcal{E}(T^2)$.

Partial solutions — 2

Let $C_\nu \subseteq C$ be the set c s.t. $(c, \nu(c)) = \nu(s) \forall s \in \nu(c)$. Let $S_\nu = \cup_{c \in C_\nu} \nu(c)$.

Let $\nu \in \mathcal{E}(T^2)$.

Proposition. *ν on $C_\nu \cup S_\nu$ is in the core of $\langle C_\nu, S_\nu, P|_{C_\nu \cup S_\nu} \rangle$.*

Proposition. *Let μ be in the core with singles, and let C' and S' denote the agents who are single in μ. If μ' is in the core with singles of $\langle C', S', P|_{C' \cup S'} \rangle$, then the matching (μ, μ'), which matches C' and S' according to μ', and $C \backslash C'$ and $S \backslash S'$ according to μ, is in the core with singles of $\langle C, S, P \rangle$.*

Restrictions on Preferences

P satisfies the *weak top-coalition property:* $\exists$ a partition $(A_1, A_2, ..., A_k)$ of agents s.t.
$\forall a \in A_1$, A_1 is a's top choice
$\forall a \in A_i$, A_i is a's top choice, within $C \cup S - A_1 - ... - A_{i-1}$

P is *respecting* if $\exists$ P_S over 2^S, and P_C over C, s.t.

1. $\forall s \in S$, $(c, S)P(s)(c, S')$ iff SP_SS'.
2. $\forall s \in S$, $(c, S)P(s)(c', S)$ iff cP_Cc'.
3. $\forall c \in C$, $SP(c)S'$ iff SP_SS'.

Proposition. *respecting* $\Rightarrow$ *weak top-coalition property.*

Restrictions on Preferences

$\langle C, S, P \rangle$ satisfies the weak top coalition prop.

Theorem. *There is a unique core matching* μ
μ *is the largest fixed point of* T^2
T^2 *algorithm finds* μ *in at most* $\lfloor k/2 \rfloor$ *steps.*

Restrictions on Preferences

Order prematchings in usual way.

Suppose T is monotone.

Proposition. $\exists$ *core matchings* $\underline{\mu}$ *and* $\overline{\mu}$ *s.t.* $\forall \nu \in \mathcal{E}(T^2)$,

$$\overline{\mu}(c) R(c) \nu(c) R(c) \underline{\mu}(c)$$

$$\underline{\mu}(s) R(s) \nu(s) R(s) \overline{\mu}(s)$$

Comparing algorithms

- T^2 algorithm: speed depends on number of iterations.
- exhaustive search: search all matchings
  e.g. with 1200 students and 9 colleges, there are
  1.233×10^{1145} matchings.

Couples

Extension of our model to matching with couples.

- Algorithm.
- New result:
  Substitutability $\Rightarrow$ Core $=$ Pairwise Stab.