Individual and collective welfare in risk-sharing with many states

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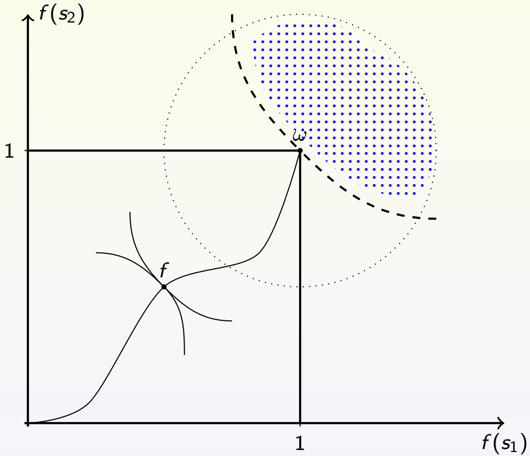
Risk sharing

Risk sharing

- An economy without aggregate risk is populated by a group of risk-averse agents.
- The agents reach an efficient risk-sharing agreement.
- Each agent i gets a state-contingent consumption plan $f_i : S \to \mathbb{R}_+$, giving utility $u_i(f_i)$.
- Then an aggregate shock arrives. The agents wish to renegotiate.
- But the status quo is sticky.
- Each agent only agrees if get at least $\varepsilon > 0$ welfare improvement.
- How likely is it that the agents can renegotiate?
- Number of states $d = |S|$ is *large*.

Risk sharing with two agents and $d = 2$



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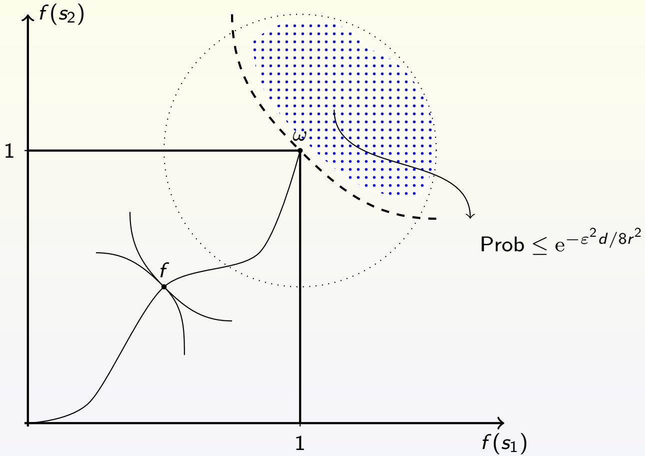

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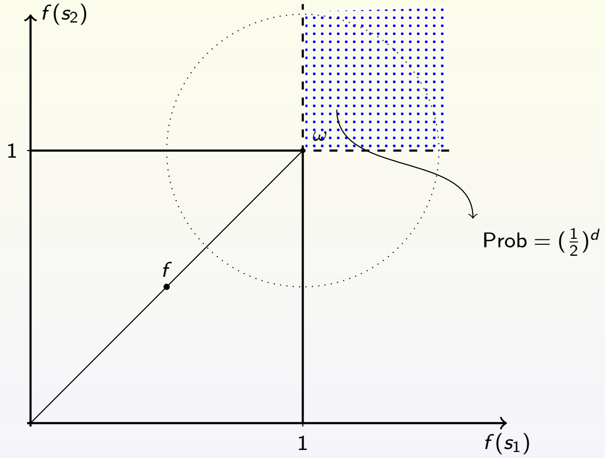


Risk sharing

What's surprising about this?

By monotonicity, $f_i + z_i \succ_i f_i$ for any $z_i > 0$.

Prob. of a (uniform) draw of $z = \sum_i z_i > 0$ is $1/2^d$.

Compare with $\mathrm{e}^{-\varepsilon^2 d/8}$.

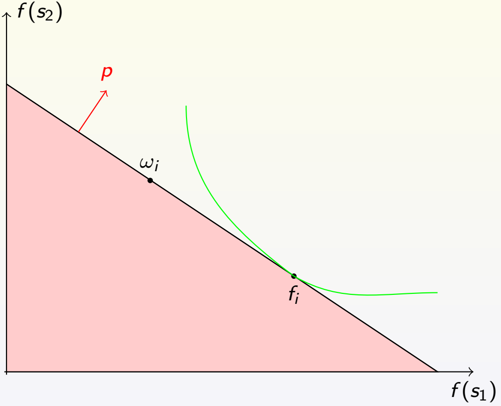
Both decrease exponentially in d.

Risk attitude doesn't matter.

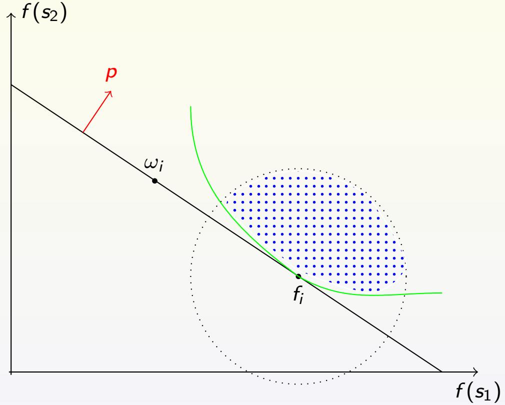
Walrasian Eq.

- Now suppose we are in a Walrasian equilibrium.
- And there is a shock.
- Can we find *one agent* (out of arbitrarily many) to absorb the shock?

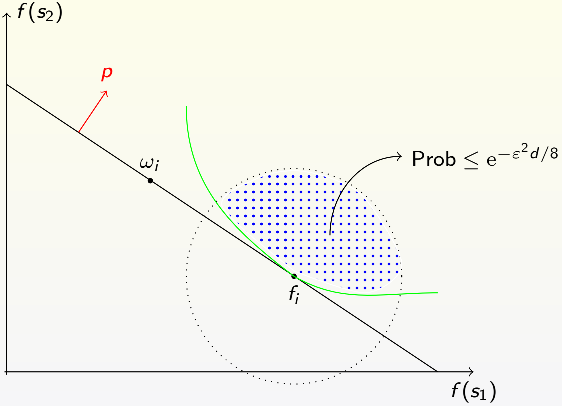
Example: Walrasian Eq with $d = 2$



Example: Walrasian Eq with $d = 2$



Example: Walrasian Eq with $d = 2$



Bound is irrespective of details of the economy. In particular, no matter what agents' preferences are.

Example: Walrasian Eq

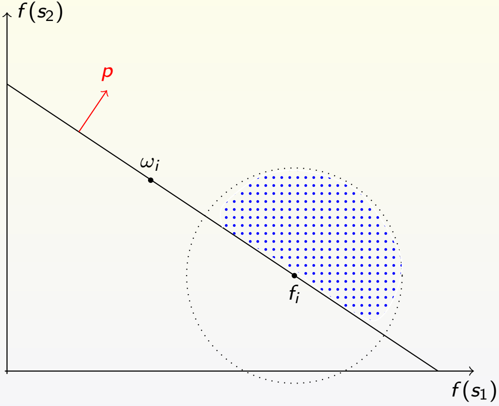
The good and the bad are separated by the budget.

The bundles that cost less are worse than f_i; and the ones that are better cost more.

The budget line divides the sphere in two equally likely subsets.

In high dimensions, however, and independently of the shape of the indifference curve the prob. of an ε-improvement shrinks to zero.

Example: Walrasian Eq with $d = 2$



We can even condition on $p \cdot z > 0$.
Same kind of result hold.

Overview of the paper

- Individual welfare in Walrasian eqm.
- Collective welfare in a PO allocation with no aggregate risk.
- Welfare and resource utilization in inefficient allocation.
- Ambiguity aversion when mutually beneficial trade is possible.
- Technique: isoperimetric inequalities and concentration of measure in high dimensions.

Definitions and notation

Let $\|\cdot\|_2$ denote the usual norm on $\mathbb{R}^d$.

Ball with center c and radius r is denoted

$$\mathbb{B}_2(c, r) = \{x \in \mathbb{R}^d : \|x - c\|_2 < r\}.$$

- If $c = 0$ we write $\mathbb{B}_2(r)$.
- If $r = 1$ we write $\mathbb{B}_2(c)$.

Economic environment: Arrow-Debreu

A finite set S of *states of the world*.

Let $d := |S|$.

An *act* is a function $f : S \to \mathbb{R}$.

We focus on *monetary acts*, $f \in \mathbb{R}^d$.

Consumption space is $\mathbb{R}^d$.

Economic environment: Preferences

Let $\succeq$ be a binary relation on $\mathbb{R}^d$.

The *weak upper contour set* of $\succeq$ at f is the set $\{g : g \succeq f\}$.

The *weak lower contour set* of $\succeq$ at f is the set $\{g : f \succeq g\}$.

Economic environment: Preferences

Let $\succeq$ be a binary relation on $\mathbb{R}^d$.

$\succeq$ is a (weakly monotone) *preference relation* if:

- (Weak Order): $\succeq$ is complete and transitive.
- (Continuity): The upper and lower contour sets are closed.
- (Monotonicity): For all $f, g \in \mathbb{R}^d$ if $f(s) \geq g(s)$ for all $s \in S$, then $f \succeq g$. If $f(s) > g(s)$ for all $s \in S$, then $f \succ g$.

The space of preference relations on $\mathbb{R}^d$ is denoted by $\mathcal{P}$.

Convex preferences

A preference $\succeq$ is *convex* if its upper contour sets are convex.

Space of convex preferences is $\mathcal{C} \subset \mathcal{P}$.

Convex preferences are very common in general equilibrium theory (existence and the second welfare thm).

Many models of decision under uncertainty feature convex preferences (MEU, variational, etc).

Approximate upper contour sets

A notion of utility improvements "with slack" is key to our results.

Definition (ε-upper contour set)

The *ε-upper contour set* of $\succeq$ at $f \in \mathbb{R}^d$ is

$$\mathcal{U}^{(\varepsilon)}_{\succeq}(f) = \left\{g \in \mathbb{R}^d : (1-\varepsilon)g \succ f\right\} .$$

So $g \in \mathcal{U}^{(\varepsilon)}_{\succeq}(f)$ when g is strictly preferred to f even when a fraction ε has been "shaved off."

Exchange economies

I a (finite) set of agents.

An *exchange economy* is a mapping $\mathcal{E} : I \to \mathcal{P} \times \mathbb{R}^d_+$.

Each agent $i \in I$ is described by a preference relation $\succeq_i$ on $\mathbb{R}^d$, and an *endowment vector* $\omega_i \in \mathbb{R}^d_+$.

An exchange economy is *convex* if each preference relation $\succeq_i$ is convex.

In an exchange economy, we use $\mathcal{U}_i^{(\varepsilon)}$ to denote the upper contour set $\mathcal{U}_{\succeq_i}^{(\varepsilon)}$.

Given an exchange economy $\mathcal{E}$, the *aggregate endowment* is $\omega := \sum_{i \in I} \omega_i$.

Walrasian equilibrium

Definition (Walrasian equilibrium)

A pair (f, p) is a *Walrasian equilibrium* if $f = \{f_i : i \in I\} \in (\mathbb{R}^d)^I$, and $p \in \mathbb{R}^d_+$ are s.t

- $g_i \succ_i f_i$ implies that $p \cdot g_i > p \cdot \omega_i$,
- and $p \cdot f_i = p \cdot \omega_i$,

for every $i \in I$; and

- $\sum_i f_i = \sum_i \omega_i$ (i.e f is an allocation; or "markets clear")

When (f, p) is a Walrasian equilibrium, we say that f is a *Walrasian equilibrium allocation.*

Main result 1: Walrasian equilibrium

Let P^r denote the uniform probability law on $\mathbb{B}_2(r)$.

Let

$$\mathcal{M}_\kappa^r := \left\{ \mathsf{P}_\kappa^r \in \Delta\left(\mathbb{B}(r)\right) : \frac{\mathrm{d}\mathsf{P}_\kappa^r}{\mathrm{d}\mathsf{P}^r} \leq \kappa \right\}$$

be the class of prob. measures supported on $\mathbb{B}(r)$ that are abs. cont. wrt P^r, and their RN derivative is bounded above by a constant $\kappa > 0$.

Ex: standard Gaussian measure conditional on $\mathbb{B}(r)$ has $\kappa \leq \mathrm{e}^{r^2/2}$.

Main result 1: Walrasian equilibrium

Theorem

Let $\mathcal{E}$ be an exch. economy. Let $\tau > 0$ s.t $\omega_i \geq \tau 1$,
and f a Walrasian eqm. allocation.
Fix $r, \varepsilon > 0$. Let $z \sim \mathrm{P}^r_\kappa \in \mathcal{M}^r_\kappa$. Then,

$$\mathrm{P}^r_\kappa\left(\,(1-\varepsilon)(f_i + \tilde{z}) \succ f_i \text{ for at least one } i \in I\,\right) \leq \kappa \mathrm{e}^{-\varepsilon^2\tau^2 d/8r^2}\,,$$

Obs 1: I could be very large. Just looking one agent in I who can take on z.
Obs 2: May condition $\widetilde{\mathrm{P}}^r_\kappa$ on $\{z : p \cdot z > 0\}$. Puts a $2\times$ on the upper bound.

NASDAQ numbers

Suppose $\tau = r$ and consider a 10% welfare improvement ($\varepsilon = 0.1$).

The (uniform) prob. of making at least one agent better of is at most $e^{-d/800}$.

Finance: d is (at least) number of real assets traded.

If d is the number of stocks trading on the NASDAQ Exchange, then bound in the thm is about 1%.

Main result 2: No aggregate uncertainty

$\mathcal{E}$ exhibits *no aggregate uncertainty* if $s \mapsto \sum_{i \in I} \omega_i(s)$ is constant.

So

$$\omega = (\bar{\omega}, \ldots, \bar{\omega}).$$

Main result 2: Scitovsky contour

Notation: Given an allocation f and $\varepsilon > 0$, let $\mathcal{V}^{(\varepsilon)} := \sum_{i \in I} \mathcal{U}_i^{(\varepsilon)}(f_i)$ be the Minkowski sum of the approximate upper contour sets.

$\mathcal{V}^{(\varepsilon)}$ is the *ε-Scitovsky contour* at f.

Theorem

Let $\mathcal{E}$ be a convex exchange economy w/no aggregate uncertainty. Normalize the agg. endow. to $\omega = 1$.
Let f be a weakly PO allocation.
Fix $r, \varepsilon > 0$. Let $z \sim \mathrm{P}_\kappa^r \in \mathcal{M}_\kappa^r$.
Then,

$$\mathrm{P}_\kappa^r \left(\sum_i f_i + \tilde{z} \in \mathcal{V}^{(\varepsilon)} \right) \leq \kappa \mathrm{e}^{-\varepsilon^2 d / 8r^2} .$$

Debreu's Coefficient of Resource Utilization

Suppose f is *not* Pareto optimal.

What is the min. aggregate resources (call it ω^*) that could provide agents with the same utility as in f?

Gap between ω and ω^* as the inefficiency inherent in the allocation f

In Debreu's words, these are "nonutilized resources."

Measure gap by a "distance with economic meaning:" $p \cdot (\omega - \omega^*)$; p is an "intrinsic price vector" associated with ω^*.

For a scale-independent measure, he works with the ratio of $p \cdot \omega^* / p \cdot \omega$.

Prices p follow argument analogous to Second Welfare Theorem.

Debreu's Coefficient of Resource Utilization

Debreu's *coefficient of resource utilization* for an allocation $f = (f_1, \ldots, f_n)$ is:

$$\mathrm{CRU}(f) := \max_{\omega^* \in \partial\overline{\mathcal{V}^{(0)}}} \frac{p(\omega^*) \cdot \omega^*}{p(\omega^*) \cdot \omega},$$

where $\partial\overline{\mathcal{V}^{(0)}}$ consists of the minimal elements of the closure $\overline{\mathcal{V}^{(0)}}$ of $\mathcal{V}^{(0)}$ (meaning there is no smaller element in $\mathcal{V}^{(0)}$), and $p(\omega^*)$ is a supporting price vector at ω^*,

Debreu's Coefficient of Resource Utilization

Proposition

Let $\mathcal{E} \colon I \to \mathcal{C} \times \mathbb{R}^d_+$ be an exchange economy under the hypotheses of Theorem 4. Fix $r > 0$ and let $z \sim \mathsf{P}^r_\kappa$. If f is not weakly Pareto optimal, and $\beta := \mathrm{CRU}(f)$ its coefficient of resource utilization, then

$$\mathsf{P}^r_\kappa \left(\omega + z \in \mathcal{V}^{(1-\beta^2)}(f) \right) \leq \mathrm{e}^{-\left(\frac{1-\beta}{\beta}\right)^2 d/8r^2} \, . \tag{1}$$

Debreu's Coefficient of Resource Utilization

Debreu: think of $\mathrm{CRU}(f)$ as a % of national income, or GDP.

But in an economy with a large state space, even a seemingly large inefficiency — as measured by CRU — may not translate into a wide scope for welfare improvements by changing aggregate consumption.

NASDAQ example: a seemingly large inefficiency of 10%, meaning $\mathrm{CRU}(f) = 0.9$, translates into a welfare improvement of 19% w/prob. bounded by 0.21%.

Main result 3: Risk sharing and Multiple-priors

Billot, Chateauneuf, Gilboa, and Tallon, 2000

Ng, 2003

Rigotti, Shannon, and Strzalecki, 2008

Gilboa, Samuelson, and Schmeidler, 2014

Ghirardato and Siniscalchi, 2018

Definitions and notation

Given a measurable subset $A \subseteq \mathbb{R}^m$, its *Euclidean volume*, denoted by $\mathrm{Vol}(A)$, is its Lebesgue measure relative to the affine hull of A.

For ex. if A is a $m-1$ dimensional surface in $\mathbb{R}^m$, then $\mathrm{Vol}(A)$ refers to the surface area of A, as opposed to its m dimensional volume (which is zero).

If S is a finite set, we denote by $\Delta S = \{\mu : S \mapsto \mathbb{R}_+ | \sum_{s \in S} \mu(s) = 1\}$ the set of all probability measures on S.

Multiple-prior preferences

Consider an exchange economy $\mathcal{E}$ with no aggregate uncertainty.

The aggregate endowment is the same across all states of the world: $\omega = (\bar{\omega}, \ldots, \bar{\omega})$. We quantify the space of all allocations, denoted by $\mathcal{F}_{\bar{\omega}}$, by the magnitude

$$\rho := 2\bar{\omega}^{-1} \max_{f \in \mathcal{F}_{\bar{\omega}}} \sum_{i \in I} \|f_i\| \,,$$

For the purposes of the talk (the paper has a more general model), suppose agents' preferences have an MEU representation:

$$u_i(f) = \min\{f \cdot \mu : \mu \in \Pi_i\},$$

for a convex compact set of priors $\Pi \subseteq \Delta S$.

Multiple prior preferences

For $J \subseteq I$, let $\Pi_J = \cap_{i \in J} \Pi_i$.

Theorem

Let $\mathcal{E}$ be an exchange economy with MEU preferences and no aggregate uncertainty.
If the allocation f is ε-Pareto dominated, then for every $J \subset I$,

$$\frac{\min\left(\mathrm{Vol}\left(\Pi_J\right), \mathrm{Vol}\left(\Pi_{J^c}\right)\right)}{\mathrm{Vol}\left(\Delta_d\right)} \leq \frac{1}{2}\,\mathrm{e}^{-c\varepsilon\sqrt{d}}\,.$$

Where $c > 0$ is a universal constant.

Multiple prior preferences

A "behavioral" analogue of small volume.

Measure degree of ambiguity aversion by the difference between max and min EU of a normalized act f ($\|f\|_2 = 1$):

$$\theta(f) := \max\{f \cdot \mu : \mu \in \Pi\} - \min\{f \cdot \mu : \mu \in \Pi\}.$$

Proposition

Under the conditions of the prev. thm, and when Π has constant width θ,

$$\theta \leq 4\, \mathrm{e}^{-c\varepsilon/\sqrt{d}} (d!)^{-1/2d}\,. \tag{2}$$

Where $c > 0$ is a universal constant $c > 0$.

Isoperimetric inequalities: some history

Relation between area/volume and shape.

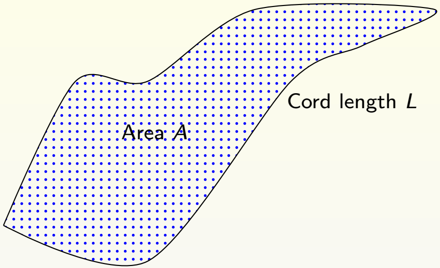
Dido Purchases Land for the Foundation of Carthage. Engraving by Matthäus Merian the Elder, in *Historische Chronica*, Frankfurt a.M., 1630. Dido's people cut the hide of an ox into thin strips and try to enclose a maximal domain.

Isoperimetric inequalities: some history

Pappus of Alexandria (On the Sagacity of Bees):

> *Bees, . . . know just this fact which is useful to them, that the hexagon is greater than the square and the triangle and will hold more honey for the same expenditure of material in constructing each. . . .*
> *We, claiming a greater share in the wisdom than the bees, will investigate a somewhat wider problem, namely that, of all equilateral and equiangular plane figures having an equal perimeter, that which has the greater number of angles is always greater, and the greatest of them all is the circle having its perimeter equal to them.*

Isoperimetric inequalities: some history



Isoperimetric ineq. on the plane:

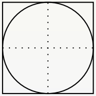
$$L^2 \geq 4\pi A$$

Equality holds iff a circle.

High-dimensional concentration of measure phenomenon.

Volume of $\mathbb{B}_2$ is $\pi^{d/2}/\Gamma(d/2+1) \sim d^{-d/2}$.

The volume of a circumscribing square is $= 2^d$.



If say $d = 20$ then chances of a random point in Square being in Ball are effectively zero.

Isoperimetric inequalities: Modern theory

High-dimensional concentration of measure phenomenon.

Let $A \subseteq \mathbb{B}_2$ have measure $\geq 1/2$.

Then the "δ-padding" of A, the set of points that are within distance δ of A, concentrates most of the meausure in $\mathbb{B}$.

Moreover, bounds on such concentration (as a function of d) are independent of A.

Notation

Let $A \subseteq \mathbb{R}^m$.

$$\text{dist}(x, A) := \inf_{a \in A} \|x - a\|$$

When a particular p-norm is used, we refer to the distance function by dist_p and the norm by $\|\cdot\|_p$.

For two subsets A and B of $\mathbb{R}^m$ we define
$\text{dist}(A, B) = \inf \{\|a - b\| : a \in A, b \in B\}$.

Notation

For a vector $p \in \mathbb{R}^d$ and a constant b, we define two half-spaces:

$$H^+(p\,;\,b) = \left\{x \in \mathbb{R}^d : p \cdot x \geq b\right\}\,,$$
$$H^-(p\,;\,b) = \left\{x \in \mathbb{R}^d : p \cdot x \leq b\right\}\,,$$

Easy to verify:

$$\mathrm{dist}_2\left(H^+(p\,;\,b_2), H^-(p\,;\,b_1)\right) = \frac{b_2 - b_1}{\|p\|_2}\,. \tag{3}$$

Isoperimetric inequalities

Let A and B be two non-empty compact subsets of $\mathbb{R}^d$.

The Brunn-Minkowski inequality is

$$\mathrm{Vol}(A + B)^{1/d} \geq \mathrm{Vol}(A)^{1/d} + \mathrm{Vol}(B)^{1/d}\,. \tag{4}$$

A *dimension-free* version of this inequality:

For $\lambda \in [0, 1]$:

$$\mathrm{Vol}(\lambda A + (1 - \lambda)B) \geq \mathrm{Vol}(A)^{\lambda}\,\mathrm{Vol}(B)^{1-\lambda}\,. \tag{5}$$

((5) may be derived as a consequence of (4))

Isoperimetric inequalities

Simple (but important) consequence of (5).

Lemma

Assume A and B are Borel subsets of $\mathbb{B}_2(r)$, and $\text{dist}_2(A, B) \geq \delta$. Then,

$$\frac{\min\{\text{Vol}(A), \text{Vol}(B)\}}{\text{Vol}(\mathbb{B}_2(r))} \leq \mathrm{e}^{-\delta^2 d/8r^2} . \tag{6}$$

Proof of the lemma

Wlog take A and B closed.

By the parallelogram law for the ℓ_2-norm if $a \in A$ and $b \in B$ then

$$\|a+b\|^2 = 2\|a\| + 2\|b\|^2 - \|a-b\|^2 \leq 4r^2 - \delta^2 \,,$$

Hence

$$\frac{A+B}{2} \subseteq \sqrt{1 - \frac{\delta^2}{4r^2}}\,\mathbb{B}(r)\,,$$

and therefore,

$$\mathrm{Vol}\left(\frac{A+B}{2}\right) \leq \left(1 - \frac{\delta^2}{4r^2}\right)^{d/2} \mathrm{Vol}(\mathbb{B}(r)) \leq \mathrm{e}^{-\delta^2 d/8r^2}\,\mathrm{Vol}(\mathbb{B}(r))\,.$$

From BM (w/ $\lambda = 1/2$) we have

$$\mathrm{Vol}\left(\frac{A+B}{2}\right) \geq \sqrt{\mathrm{Vol}\,(A)}\sqrt{\mathrm{Vol}\,(B)} \geq \min\{\mathrm{Vol}\,(A)\,, \mathrm{Vol}\,(B)\}$$

Proof of first thm

Given f is a Walrasian eq. there's $p \in \mathbb{R}^d_+$ s.t $p \cdot g_i > p \cdot \omega_i$ for all $i \in I$ and $g_i \in \mathcal{U}_i^{(0)}(f_i)$.

Observe that if $g \in \mathcal{U}_i^{(\varepsilon)}(f_i)$ then $(1-\varepsilon)g \in \mathcal{U}_i^{(0)}(f_i)$ and therefore $p_i \cdot (1-\varepsilon)(g-\omega_i) > \varepsilon p \cdot \omega_i$.

So:

$$p \cdot (g - \omega_i) > \frac{\varepsilon p \cdot \omega_i}{1-\varepsilon} > \varepsilon p \cdot \omega_i \geq \varepsilon \tau \|p\|_1 ,$$

Hence, $\mathcal{U}_i^{(\varepsilon)}(f_i) - \{\omega_i\} \subseteq H^+(p; \varepsilon\tau\|p\|_1)$ for all $i \in I$.

Proof of first thm

Define $\mathcal{Q} = \bigcup_{i \in I} \left(\mathcal{U}_i^{(\varepsilon)}(f_i) - \{\omega_i\} \right)$.

Then $\mathcal{Q} \subseteq H^+ \left(p; \varepsilon\tau \|p\|_1\right)$, so

$$
\begin{aligned}
\operatorname{dist}_2 \left(\mathcal{Q} \cap \mathbb{B}_2(r), H^-(p;0) \cap \mathbb{B}_2(r) \right) \geq & \operatorname{dist}_2 \left(H^+ \left(p; \varepsilon\tau \|p\|_1\right) \cap \mathbb{B}_2(r), \right. \\
& \left. H^-(p;0) \cap \mathbb{B}_2(r) \right) \\
& \geq \operatorname{dist}_2 \left(H^+ \left(p; \varepsilon\tau \|p\|_1\right), H^-(p;0) \right) \\
& = \varepsilon\tau \frac{\|p\|_1}{\|p\|_2} \\
& \geq \varepsilon\tau
\end{aligned}
$$

Now set $A := \mathcal{Q} \cap \mathbb{B}_2(r)$ and $B := H^-(p;0) \cap \mathbb{B}_2(r)$.

The above shows $\operatorname{dist}_2(A, B) \geq \varepsilon\tau$. But B covers at least $1/2$ vol. of $\mathbb{B}_2(r)$.

Proof of first thm

So must have $\mathsf{Vol}(A) \leq \mathsf{Vol}(B)$.

The lemma implies $\mathsf{Vol}(A)/\mathsf{Vol}(\mathbb{B}_2(r)) \leq \mathrm{e}^{-\varepsilon^2\tau^2 d/8r^2}$.

So

$$\frac{\mathsf{Vol}\left(\mathcal{Q} \cap \mathbb{B}_2(r)\right)}{\mathsf{Vol}(\mathbb{B}_2(r))} \leq \mathrm{e}^{-\varepsilon^2\tau^2 d/8r^2},$$

Proof of first thm

Finally note if $f = \{f_i : i \in I\}$ is a Walrasian eq. for the exchange economy $\mathcal{E}$, it's also one for $\mathcal{E}'$ that's identical to $\mathcal{E}$ except that $\omega_i' = f_i$.

Conclusion

A random perturbation of individual, or collective, consumption may improve welfare.

But the probability that this occurs by a fixed amount ε decreases exponentially in the number of states.

Applications to: CRU (in lieu of ε) and ambiguity aversion.

Arguments follow from high-dimensional probability phenomena that have been the focus of a recent active literature.