Efficiency and Fairness in Random Resource Allocation and Social Choice

Federico Echenique  
Caltech

Joseph Root  
U. Chicago

Fedor Sandomirskiy  
Caltech

Universidad de Chile

May 11, 2022
A (very) simple example

- Two outcomes: $a$ and $b$
- Two agents 1 and 2
  - $a \succ_1 b$ and $b \succ_2 a$
- How to choose fairly?
  - Answer: flip a coin $\frac{1}{2}a$ and $\frac{1}{2}b$
  - Intuitively fair and efficient
- Note: must view the coin-flip *ex ante*.
- Ex-post the coin-flip is still unfair.
Another example

- Six outcomes: $a, b, c, x, y, z$.
- Three agents: $1, 2, 3$.

\[
\begin{align*}
a_1 &\succ_1 x_1 \succ_1 b_1 \succ_1 y_1 \succ_1 c_1 \succ_1 z_1 \\
b_2 &\succ_2 z_2 \succ_2 c_2 \succ_2 x_2 \succ_2 a_2 \succ_2 y_2 \\
c_3 &\succ_3 y_3 \succ_3 a_3 \succ_3 z_3 \succ_3 b_3 \succ_3 x_3
\end{align*}
\]

- Two interwoven Condorcet cycles
- Again all agents “symmetric”
- All outcomes are Pareto efficient. How to choose fairly?
- Consider the uniform lottery $\frac{1}{6}a, \frac{1}{6}x, \frac{1}{6}b, \frac{1}{6}y, \frac{1}{6}c$ and $\frac{1}{6}z$
- And the alternative lottery $\frac{1}{3}a, \frac{1}{3}b$ and $\frac{1}{3}c$. 

Echenique, Root, Sandomirskiy

Random Social Choice
Another example

Echenique, Root, Sandomirskiy Random Social Choice
Another example

\begin{align*}
\frac{1}{3} \\
\frac{1}{6}
\end{align*}

\begin{align*}
 a & \quad x & \quad b & \quad y & \quad c & \quad z
\end{align*}
Another example

So all agents prefer

\[
\frac{1}{3}a, \frac{1}{3}b, \frac{1}{3}c
\]

to

\[
\frac{1}{6}a, \frac{1}{6}x, \frac{1}{6}b, \frac{1}{6}y, \frac{1}{6}c, \frac{1}{6}z.
\]

Note contrast with our first example.
Takeaways

- Randomization is required for fairness.
- As long as it’s evaluated ex-ante.
- Ex-ante, though, a randomization among efficient outcomes may not be efficient.
- Problem: the Pareto frontier may not be convex.
Questions

- How peculiar is the example? I.e. when is the Pareto frontier convex?
- What is the meaning of fairness for random outcomes?
- Can fairness and efficiency be achieved?
Our Contributions

- How peculiar is the example? I.e. when is the Pareto frontier convex? A new characterization
- What is the meaning of fairness for random outcomes? A simple new fairness notion, formalizing the “symmetries” above
- Can fairness and efficiency be achieved? An existence result and a procedure
Literature Review

- Randomized Social Choice
  - Gibbard (1977); Fishburn (1984); Carroll (2010); Aziz, Brandl and Brandt (2015); Aziz, Brandl, Brandt and Brill (2018)

- House Allocation

- Dichotomous Domains
  - Bogomolnaia, Moulin and Strong (2002); Duddy (2015)
Model primitives:

\[(N, X, (\succsim_i)_{i \in N})\]

- Finite set of agents, \(N\).
- Finite set of outcomes, \(X\).
- A *preference* on \(X\) is a weak order (complete and transitive binary relation).
- Each agent \(i \in N\) is endowed w/a preference \(\succsim_i\) on \(X\).
Applications

1. Arrovian social choice (i.e strict preferences)
2. approval voting (dichotomous domains)
3. single-peaked preferences
4. house allocation
5. school choice
Let $r_i(k)$ be the set of outcomes in $i$’s top $k$ indifference classes.

For example

$$r_i(1) = \{x \in X : x \succeq_i y \text{ for all } y \in X\}$$

and

$$r_i(2) = r_i(1) \cup \{x \in X : x \succeq_i y \text{ for all } y \in X \setminus r_i(1)\}.$$

So $x \succ_i y$ iff there is $k$ with $x \in r_i(k)$ and $y \notin r_i(k)$. 
Let $\Delta(X)$ be the set of probability distributions on $X$. We call the elements of $\Delta(X)$ *lotteries*.

We extend $\succeq_i$ to an ordering $\succeq_{SD}^i$ on $\Delta(X)$ by:

\[ p \succeq_{SD}^i q \text{ iff } \sum_{j \in r_i(k)} p_j \geq \sum_{j \in r_i(k)} q_j \text{ for all } k \]

And $p \succ_{SD}^i q$ when $p \succeq_{SD}^i q$ and at least one of the defining inequalities is strict.
That is, $p \succeq_{SD}^{i} q$ iff $p$ first-order stochastically dominates $q$ according to $\succeq_{i}$.

Equivalently, for all VNM utility $u_i$ that represents $\succeq_{i}$, $u_i(p) \geq u_i(q)$.

I will abuse notation and use $\succeq_{i}$ to mean $\succeq_{SD}^{i}$. 
An outcome \( x \in X \) is *efficient* (or *Pareto optimal*) if there is no outcome \( y \in X \) with \( y \preceq_i x \) for all \( i \in N \) and \( y >_i x \) for some \( i \in N \).

Let \( X^* \) denote all the efficient outcomes in \( X \).

A lottery \( p \in \Delta(X) \) is *ex-post efficient* if \( p(X^*) = 1 \).
A lottery $p$ is \textit{ex-ante efficient} if there is no lottery $q$ s.t $q \succsim_i p$ for all $i \in N$ and $q \succ_i p$ for some $i \in N$.

Obs: Ex-ante efficiency implies ex-post efficiency
Questions

- How peculiar is the example? I.e. when is the Pareto frontier convex?
- What is the meaning of fairness for random outcomes?
- Can fairness and efficiency be achieved?
Some special cases

**Proposition**

Every ex-post efficient lottery is ex-ante efficient whenever the preference profile \((\succsim_i)_{i \in N}\)

1. has \(|X^*| \leq 3\) Pareto optimal outcomes.
2. has \(|N| \leq 2\) agents.
3. has \(|N| = 3\) agents and \(|X^*| \leq 5\) Pareto optimal outcomes.

Moreover, for any pair of \(|X^*|\) and \(|N|\) which do not satisfy one of these conditions, there is a preference profile where ex-ante and ex-post differ.
For any two outcomes $\alpha$ and $\beta$, some agents prefer $\alpha$ and some prefer $\beta$.

This is not true with subsets of outcomes.

$\{a, b, c\}$ is in some sense preferred to $\{x, y, z\}$ by everyone.

For every outcome $o$, and agent $i$, the set $\{a, b, c\}$ has a greater intersection with the upper contour set of $o$ than $\{x, y, z\}$.
An example with dichotomous preferences

- Five agents: 1, 2, 3, 4, 5.
- Four outcomes: $a, b, c, d$.
- Agents either approve or disapprove of each outcome.

<table>
<thead>
<tr>
<th></th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

- All outcomes Pareto efficient
- The uniform lottery is dominated by $\frac{1}{2}a + \frac{1}{2}b$.
- The set $\{a, b\}$ dominates the set $\{c, d\}$
The general pattern

- These examples gave us the general pattern with one exception
- The sets might need to have repetitions

**Theorem**

The (ex-ante) Pareto frontier is convex iff one cannot find a pair of multisets $A = \{a_k\}_{k=1,\ldots,K}$ and $B = \{b_k\}_{k=1,\ldots,K}$ consisting of outcomes from $X^*$ such that each outcome is repeated at most $\exp\left(\frac{1}{2} |X^*| \cdot \ln |X^*| \right)$ times and

$$|\{k : a_k \succ_i x\}| \geq |\{k : b_k \succ_i x\}|$$

for any $x \in X^*$ and $i$ and for at least one combination, the inequality is strict.
Proof outline

Proof strategy:
1. Convert the problem into a linear system
2. Show the system has integral solutions
3. Deduce the multiset representation
Lemma

The ex-ante frontier is convex iff there is no $\alpha \in \mathbb{R}^{X^*}$ such that

$$\sum_{x \in X^*} \alpha_x = 0,$$

$$\sum_{y \succ_i x} \alpha_y \geq 0 \quad \text{for all } i \in N \text{ and } x \in X^*$$

and at least one of these inequalities is strict.

proof outline:

- Suppose $p$ is supported on $X^*$ and is dominated by $q$. Let $u$ be the uniform lottery over $X^*$.
- Then $u = \lambda p + (1 - \lambda)y$, where $y = \frac{1}{1-\lambda}(u - \lambda p)$.
proof outline continued:

- If \( p \) is dominated by \( q \) then \( u = \lambda p + (1 - \lambda)y \) is dominated by \( \lambda q + (1 - \lambda)y \).
- The ex-ante frontier is convex \( \iff \) the uniform lottery over \( X^* \) is ex-ante efficient.
- If \( q \) dominates the uniform lottery, set \( \alpha = q - u \).
- Conversely, if there is some nontrivial solution \( \alpha \), let \( q = u + \varepsilon \alpha \) for \( \varepsilon > 0 \) small.
Now, we have a homogeneous system $Ax \succeq 0$ with all entries in $A$ either 0 or 1.

Convexity of the Pareto frontier $\iff$ the only solution is 0.

The set of solutions to this system is a cone.

To get an explicit solution, we can add the constraints that $-1 \leq x_i \leq 1$ for all $i$.

This gives a polytope $P = A'x \succeq b$. At least one nonzero extreme point $v$. 
Proof outline

- $A''v = b''$ where $A''$ is $|X^*|$ independent rows of $A'$
- By Cramer’s rule $v$ has rational entries. Get rid of the denominators
- Let $A$ be the multiset where each $x$ such that $v_x > 0$ appears $v_x$ times
- Let $B$ be the multiset where each $y$ such that $v_y < 0$ appears $v_y$ times
Some specific settings

Corollary

The ex-ante Pareto frontier is **not** convex in the following settings

- House allocation (Bogomolnaia and Moulin (2001))
- Arrovian social choice
- Two-sided matching
- Dichotomous domains (Brandt et al (2016))

The ex-ante Pareto frontier **is** convex in the following setting

- Single-peaked preferences
Questions

- How peculiar is this example? I.e. when is the Pareto frontier convex?
- **What is the meaning of fairness for random outcomes?**
- Can fairness and efficiency be achieved?
Symmetry – an example

- $a \succ_1 b$ and $b \succ_2 a$
- The labels $a$ and $b$ don’t have meaning
- Could rename $a \rightarrow b$ and $b \rightarrow a$
- Now the preference is $b \succ_1 a$ and $a \succ_2 b$
- The names of the agents also don’t have any meaning
- Could rename $1 \rightarrow 2$ and $2 \rightarrow 1$ to get back $a \succ_1 b$ and $b \succ_2 a$
- 1 and 2 only differ in the labels we assign to agents and objects
Symmetry – another example

\[
\begin{align*}
    &a \succ_1 b \succ_1 c \\
    &b \succ_2 c \succ_2 a \\
    &c \succ_3 a \succ_3 b \\
\end{align*}
\]

Permutation:

\[
\sigma = \begin{pmatrix}
    a & b & c \\
    b & c & a \\
\end{pmatrix}
\]

Then:

\[
\begin{align*}
    &b \succ_1^\sigma c \succ_1^\sigma a \\
    &c \succ_2^\sigma a \succ_2^\sigma b \\
    &a \succ_3^\sigma b \succ_3^\sigma c \\
\end{align*}
\]

Agent 1 is to this economy what 2 was to the first economy.
Symmetries of preference profiles

Given \( \succsim_i \) on \( X \) and a permutation \( \sigma : X \rightarrow X \), can form \( \succsim_i^\sigma \) s.t.

\[
x \succsim_i y \iff \sigma(x) \succsim_i \sigma(y).
\]

Consider the group \( \Sigma_n \times \Sigma_m \) of pairs of permutations \((\sigma, \nu)\), with \( \sigma \) over \( X \) and \( \nu \) over \( N \).

Think of \((\sigma, \nu)\) as operating on preference profiles: \((\succsim_i)_{i \in N} \mapsto (\succsim_{\nu(i)})_{i \in N}\)

For a profile \( \succsim = (\succsim_i)_{i \in N} \), let \( Z \subseteq \Sigma_n \times \Sigma_m \) be the subgroup that leaves \( \succsim \) fixed: the \textit{stabilizer subgroup}. 

For the previous example

\[ a \succ_1 b \succ_1 c \]
\[ b \succ_2 c \succ_2 a \]
\[ c \succ_3 a \succ_3 b \]

\[ \sigma = \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}, \quad \nu = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \]

\[ Z = \{(\sigma, \nu), \text{Id}, (\sigma^{-1}, \nu^{-1})\} \]
A **symmetry** of a preference profile \((\succeq_i)_{i \in \mathbb{N}}\) is a pair \((\sigma, \nu) \in \mathbb{Z}\).

So
\[
(\succeq_{\sigma(1)}, \succeq_{\sigma(2)}, \ldots, \succeq_{\sigma(n)}) = (\succeq_i)_{i \in \mathbb{N}}
\]

Agents \(i\) and \(j\) are **symmetric** if there is a symmetry \((\sigma, \nu)\) s.t. \(\nu(i) = j\).

Obs in that case, \((\sigma^{-1}, \nu^{-1})\) is also a symmetry and \(i = \nu^{-1}(j)\). So notion of symmetry is indeed symmetric.
We also have \((\sigma, \nu)\) operating on lotteries: \(q \mapsto q^\sigma\).

The *orbit* of a lottery \(q\) is the set \(O(q)\) obtained by the repeated application of permutations in \(Z\).

Since \(Z\) is a group, this is the same as

\[
O(q) = \{q^\sigma : (\sigma, \nu) \in Z\}.
\]
A new notion of fairness

A lottery \( p \) satisfies *equal treatment of symmetric agents (ETOSA)* if for any symmetric agents \( i \) and \( j \),

\[
\sum_{x \in r_i(k)} p_x = \sum_{y \in r_j(k)} p_y
\]

for all \( k \).

Obs: the uniform lottery on \( X \) satisfies ETOSA, as \( |r_i(k)| = |r_j(k)| \) when \( i \) and \( j \) are symmetric.
- Well defined for any social choice environment: Arrovian, house allocation, etc.
- A fixed-profile notion.
- Doesn’t require a mechanism, or counterfactual choices.
- In house allocation, strictly stronger than equal treatment of equals
A new notion of fairness – back to the example

\[ a \succ_1 x \succ_1 b \succ_1 y \succ_1 c \succ_1 z \]
\[ b \succ_2 z \succ_2 c \succ_2 x \succ_2 a \succ_2 y \]
\[ c \succ_3 y \succ_3 a \succ_3 z \succ_3 b \succ_3 x \]

- 1 and 2 are symmetric under the symmetry \((\sigma, \nu)\) where

\[ \sigma = \begin{pmatrix} a & x & b & y & c & z \\ b & z & c & x & a & y \end{pmatrix} \quad \text{and} \quad \nu = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \]
A new notion of fairness – back to the example

- All agents are symmetric  \( \implies p_a = p_b = p_c \) and \( p_x = p_y = p_z \).
- Recall the uniform lottery \( \frac{1}{6} a, \frac{1}{6} x, \frac{1}{6} b, \frac{1}{6} y, \frac{1}{6} c \) and \( \frac{1}{6} z \)
- And the alternative \( \frac{1}{3} a, \frac{1}{3} b \) and \( \frac{1}{3} c \)
- Both satisfy ETOSA, but one is ex-ante efficient and the other is not.
Questions

- How peculiar is the example? I.e. when is the Pareto frontier convex?
- What is the meaning of fairness for random outcomes?
- Can fairness and efficiency be achieved?
Proposition

Let $p$ be any lottery that satisfies ETOSA. There is an ex-ante efficient lottery which dominates $p$ and satisfies ETOSA.
How to improve on inefficient but fair outcomes

Notation:

\[ \sum_{x \in A} p(x) = p(A) \]

Let \( p \) satisfy ETOSA but be inefficient.

Let \( q \) Pareto dominate \( p \).

Fix \( (\sigma, \nu) \) a symmetry of \( (\succeq_i)_{i \in N} \) and \( i = \nu(j) \).

Then \( x \in r_i(k) \) iff \( \sigma(x) \in r_j(k) \).
How to improve on inefficient but fair outcomes

Define $q^\sigma$ by $q^\sigma(\sigma(x)) = q(x)$.

Then

$$q(A) = q^\sigma(\{\sigma(x) : x \in A\})$$

$$\implies q(r_i(k)) = q^\sigma(\{\sigma(x) : x \in r_i(k)\}) = q^\sigma(r_j(k))$$

So:

$$p(r_j(k)) = p(r_i(k)) \leq q(r_i(k)) = q^\sigma(r_j(k))$$

ETOSA

$\leq$

$q$ dom. $p$

And thus $q^\sigma$ also dominates $p$. 

Echenique, Root, Sandomirskiy Random Social Choice
How to improve on inefficient but fair outcomes

Now consider the *orbit*

\[ O(q) = \{ q^\sigma : (\sigma, \nu) \in Z \} \]

We have seen that if \( p \) is inefficient but fair, then for any dominating \( q \) and \( (\sigma, \nu) \in Z \), \( q^\sigma \) also dominates \( p \).

So all the elements of \( O(q) \) dominate \( p \). Let

\[ \bar{q} = \frac{1}{|O(q)|} \sum_{\hat{q} \in O(q)} \hat{q} \]

be the average of \( O(q) \).

\( \bar{q} \) is a “symmetrization” of \( q \).
How to improve on inefficient but fair outcomes

Claim: $\bar{q}$ is ETOSA and dominates $p$.

For any VNM utility $u_i$ for $\succeq_i$, $u_i \cdot p \leq u_i \cdot q^\sigma$. Thus $u_i \cdot p \leq u_i \cdot \bar{q}$.

Moreover, if $i$ and $j$ are symmetric, then $\succeq_i = \succeq_j$, and we have seen that $\hat{q}(r_i(k)) = \hat{q}^\sigma(r_j(k))$, for $(\sigma, \nu) \in Z$.

In consequence

$$\bar{q}(r_i(k)) = \frac{1}{|O(q)|} \sum_{\hat{q} \in O(q)} \hat{q}(r_i(k)) = \frac{1}{|O(q)|} \sum_{\hat{q} \in O(q)} \hat{q}^\sigma(r_j(k)) = \bar{q}(r_j(k))$$

as $O(q) = \{ \hat{q}^\sigma : \hat{q} \in O(q) \}$ ($\hat{q} \mapsto \hat{q}^\sigma$ is a bijection).
So we’ve found a lottery $\bar{q}$ that Pareto dominates $p$ and satisfies ETOSA.

If $\bar{q}$ is not efficient, rinse and repeat.

Iterating, we should find an efficient lottery that satisfies ETOSA and dominates $p$. 
Proof

Let $P$ be the set of lotteries that weakly Pareto dominate $p$ and satisfy ETOSA.

Obs: $P$ is a polytope. It is defined by a finite set of linear inequalities:

1. $q(r_i(k)) \geq p(r_i(k))$ for all $i \in N$ and $1 \leq k \leq |X|$.
2. $q(r_i(k)) = q(r_i(k))$ for all symmetric $i$ and $j$ and $1 \leq k \leq |X|$.
3. $q \in \Delta(X)$.

And $P$ is nonempty as it contains $\bar{q}$. 
Proof

Let \( q^* \in \arg\max \{ \sum_i u_i(q) : q \in P \} \) for some choice of VNM \( u_i \) for \( \succeq_i \).

Since \( q^* \in P \) it satisfies ETOSA and dominates \( p \). Indeed by def. \( q^* \) is not dominated by any lottery in \( P \).

We claim that \( q^* \) is efficient.

For if is it not, then it is st. dominated by a lottery \( q^{**} \). Starting from \( q^{**} \) we can again compute the orbit and take the mean, yielding a lottery that satisfies ETOSA and thus dominates \( q^* \) in \( P \). This is absurd.