Efficiency and Fairness in Random Resource Allocation and Social Choice

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A (very) simple example

- Two outcomes: a and b
- Two agents 1 and 2
- $a \succ_1 b$ and $b \succ_2 a$
- How to choose fairly?
- Answer: flip a coin $\frac{1}{2}a$ and $\frac{1}{2}b$
- Intuitively fair and efficient
- Note: must view the coin-flip *ex ante*.
- Ex-post the coin-flip is still unfair.

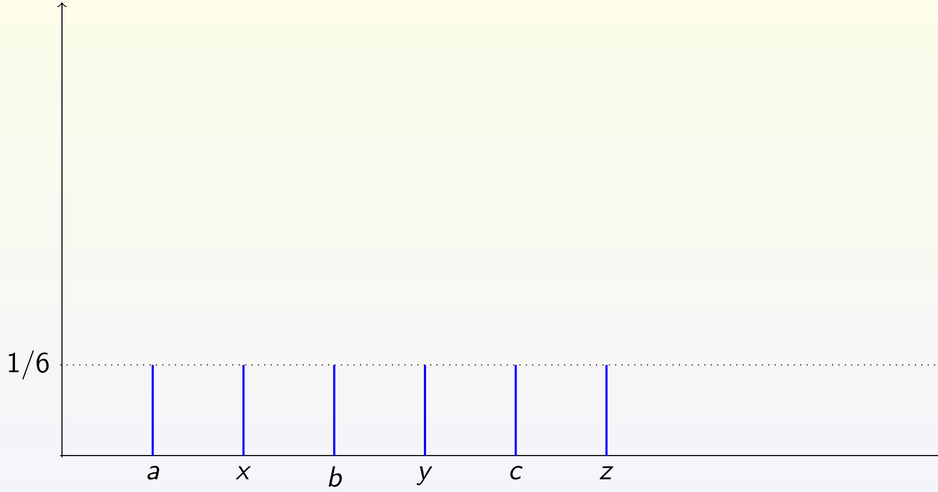
Another example

- Six outcomes: a, b, c, x, y, z.
- Three agents: $1, 2, 3$.

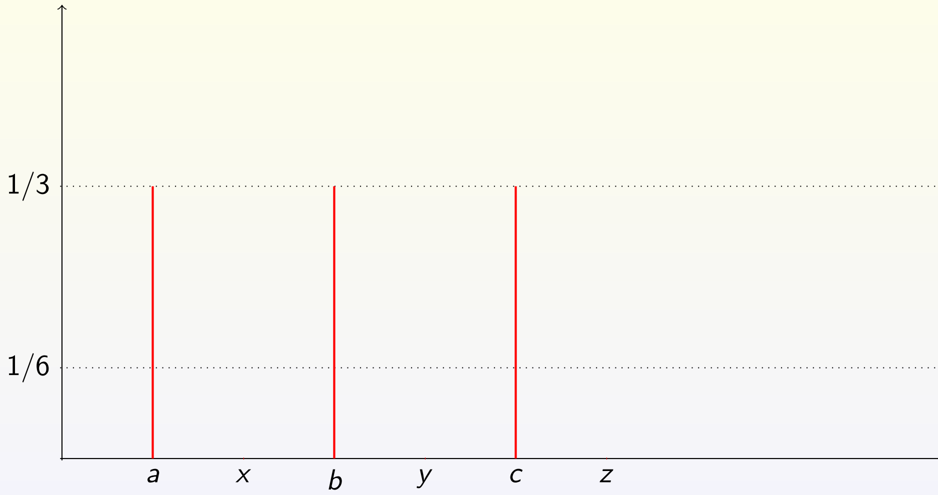
$$a \succ_1 x \succ_1 b \succ_1 y \succ_1 c \succ_1 z$$
$$b \succ_2 z \succ_2 c \succ_2 x \succ_2 a \succ_2 y$$
$$c \succ_3 y \succ_3 a \succ_3 z \succ_3 b \succ_3 x$$

- Two interwoven Condorcet cycles
- Again all agents "symmetric"
- All outcomes are Pareto efficient. How to choose fairly?
- Consider the uniform lottery $\frac{1}{6}a$, $\frac{1}{6}x$, $\frac{1}{6}b$, $\frac{1}{6}y$, $\frac{1}{6}c$ and $\frac{1}{6}z$
- And the alternative lottery $\frac{1}{3}a$, $\frac{1}{3}b$ and $\frac{1}{3}c$.

Another example



Another example



Another example

So all agents prefer

$$\frac{1}{3}a, \frac{1}{3}b, \frac{1}{3}c$$

to

$$\frac{1}{6}a, \frac{1}{6}x, \frac{1}{6}b, \frac{1}{6}y, \frac{1}{6}c, \frac{1}{6}z.$$

Note contrast with our first example.

Takeaways

- Randomization is required for fairness.
- As long as it's evaluated ex-ante.
- Ex-ante, though, a randomization among efficient outcomes may not be efficient.
- Problem: the Pareto frontier may not be convex.

Questions

- How peculiar is the example? I.e. when is the Pareto frontier convex?
- What is the meaning of fairness for random outcomes?
- Can fairness and efficiency be achieved?

Our Contributions

- How peculiar is the example? I.e. when is the Pareto frontier convex? A new characterization
- What is the meaning of fairness for random outcomes? A simple new fairness notion, formalizing the "symmetries" above
- Can fairness and efficiency be achieved? An existence result and a procedure

Literature Review

- Randomized Social Choice
  - Gibbard (1977); Fishburn (1984); Carroll (2010); Aziz, Brandl and Brandt (2015); Aziz, Brandl, Brandt and Brill (2018)
- House Allocation
  - Bogolmonaia and Moulin (2001), Mclennan (2002), Abdulkadiroğlu and Sönmez (2003)
- Dichotomous Domains
  - Bogomolnaia, Moulin and Strong (2002); Duddy (2015)

Model

Model primitives:

$$(N, X, (\succsim_i)_{i \in N})$$

- Finite set of agents, N.
- Finite set of outcomes, X.
- A *preference* on X is a weak order (complete and transitive binary relation).
- Each agent $i \in N$ is endowed w/a preference $\succsim_i$ on X.

Applications

1. Arrovian social choice (i.e strict preferences)
2. approval voting (dichotomous domains)
3. single-peaked preferences
4. house allocation
5. school choice

Preliminaries

Let $r_i(k)$ be the set of outcomes in i's top k indifference classes

For example

$$r_i(1) = \{x \in X : x \succsim_i y \text{ for all } y \in X\}$$

and

$$r_i(2) = r_i(1) \cup \{x \in X : x \succsim_i y \text{ for all } y \in X \setminus r_i(1)\}.$$

So $x \succ_i y$ iff there is k with $x \in r_i(k)$ and $y \notin r_i(k)$.

Preliminaries

Let $\Delta(X)$ be the set of probability distributions on X.
We call the elements of $\Delta(X)$ *lotteries*.

We extend $\succsim_i$ to an ordering $\succsim_i^{SD}$ on $\Delta(X)$ by:

$p \succsim_i^{SD} q$ iff

$$\sum_{j \in r_i(k)} p_j \geq \sum_{j \in r_i(k)} q_j \text{ for all } k$$

And $p \succ_i^{SD} q$ when $p \succsim_i^{SD} q$ and at least on of the defining inequalities is strict.

Preliminaries

That is, $p \succsim_i^{SD} q$ iff p first-order stochastically dominates q according to $\succsim_i$

Equivalently, for all VNM utility u_i that represents $\succsim_i$, $u_i(p) \geq u_i(q)$

I will abuse notation and use $\succsim_i$ to mean $\succsim_i^{SD}$

Background: ex-post efficiency

An outcome $x \in X$ is *efficient* (or *Pareto optimal*) if there is no outcome $y \in X$ with $y \succsim_i x$ for all $i \in N$ and $y \succ_i x$ for some $i \in N$.

Let X^* denote all the efficient outcomes in X.

A lottery $p \in \Delta(X)$ is *ex-post efficient* if $p(X^*) = 1$.

Background: ex-ante efficiency

A lottery p is *ex-ante efficient* if there is no lottery q s.t $q \succsim_i p$ for all $i \in N$ and $q \succ_i p$ for some $i \in N$.

Obs: Ex-ante efficiency implies ex-post efficiency

Questions

- How peculiar is the example? I.e. when is the Pareto frontier convex?
- What is the meaning of fairness for random outcomes?
- Can fairness and efficiency be achieved?

Some special cases

Proposition

Every ex-post efficient lottery is ex-ante efficient whenever the preference profile $(\succsim_i)_{i \in N}$

1. has $|X^*| \leq 3$ Pareto optimal outcomes.
2. has $|N| \leq 2$ agents.
3. has $|N| = 3$ agents and $|X^*| \leq 5$ Pareto optimal outcomes.

Moreover, for any pair of $|X^*|$ and $|N|$ which do not satisfy one of these conditions, there is a preference profile where ex-ante and ex-post differ

Back to the example

$$a \succ_1 x \succ_1 b \succ_1 y \succ_1 c \succ_1 z$$
$$b \succ_2 z \succ_2 c \succ_2 x \succ_2 a \succ_2 y$$
$$c \succ_3 y \succ_3 a \succ_3 z \succ_3 b \succ_3 x$$

- For any two outcomes α and β, some agents prefer α and some prefer β
- This is not true with subsets of outcomes
- $\{a, b, c\}$ is in some sense preferred to $\{x, y, z\}$ *by everyone*.
- For every outcome o, and agent i, the set $\{a, b, c\}$ has a greater intersection with the upper contour set of o than $\{x, y, z\}$

An example with dichotomous preferences

- Five agents: $1, 2, 3, 4, 5$.
- Four outcomes: a, b, c, d.
- Agents either approve or disapprove of each outcome.

	a	b	c	d
1	1	0	1	0
2	0	1	0	1
3	1	0	0	1
4	1	0	0	0
5	0	1	1	0

- All outcomes Pareto efficient
- The uniform lottery is dominated by $\frac{1}{2}a + \frac{1}{2}b$.
- The set $\{a, b\}$ dominates the set $\{c, d\}$

The general pattern

- These examples gave us the general pattern with one exception
- The sets might need to have repetitions

Theorem

The (ex-ante) Pareto frontier is convex iff one cannot find a pair of multisets $A = \{a_k\}_{k=1,\dots,K}$ and $B = \{b_k\}_{k=1,\dots,K}$ consisting of outcomes from X^* such that each outcome is repeated at most $\exp\left(\frac{1}{2}|X^*| \cdot \ln |X^*|\right)$ times and

$$|\{k : a_k \succ_i x\}| \geq |\{k : b_k \succ_i x\}|$$

for any $x \in X^*$ and i and for at least one combination, the inequality is strict

Proof outline

Proof strategy:

1. Convert the problem into a linear system
2. Show the system has integral solutions
3. Deduce the multiset representation

Proof outline

Lemma

The ex-ante frontier is convex iff there is no $\alpha \in \mathbb{R}^{X^*}$ such that $\sum_{x \in X^*} \alpha_x = 0$,

$$\sum_{y \succ_i x} \alpha_y \geq 0 \qquad \text{for all } i \in N \text{ and } x \in X^*$$

and at least one of these inequalities is strict.

proof outline:

- Suppose p is supported on X^* and is dominated by q. Let u be the uniform lottery over X^*.
- Then $u = \lambda p + (1-\lambda)y$, where $y = \frac{1}{1-\lambda}(u - \lambda p)$.

Proof outline

proof outline continued:

- If p is dominated by q then $u = \lambda p + (1 - \lambda)y$ is dominated by $\lambda q + (1 - \lambda)y$.
- The ex-ante frontier is convex $\iff$ the uniform lottery over X^* is ex-ante efficient.
- If q dominates the uniform lottery, set $\alpha = q - u$.
- Conversely, if there is some nontrivial solution α, let $q = u + \varepsilon\alpha$ for $\varepsilon > 0$ small.

Proof outline

- Now, we have a homogeneous system $Ax \geqq 0$ with all entries in A either 0 or 1
- Convexity of the Pareto frontier $\iff$ the only solution is 0
- The set of solutions to this system is a cone
- To get an explicit solution, we can add the constraints that $-1 \leq x_i \leq 1$ for all i
- This gives a polytope $P = A'x \geqq b$. At least one nonzero extreme point v

Proof outline

- $A''v = b''$ where A'' is $|X^*|$ independent rows of A'
- By Cramer's rule v has rational entries. Get rid of the denominators
- Let A be the multiset where each x such that $v_x > 0$ appears v_x times
- Let B be the multiset where each y such that $v_y < 0$ appears v_y times

Some specific settings

Corollary

The ex-ante Pareto frontier is **not** convex in the following settings

- House allocation (Bogomolnaia and Moulin (2001))
- Arrovian social choice
- Two-sided matching
- Dichotomous domains (Brandt et al (2016))

The ex-ante Pareto frontier **is** convex in the following setting

- Single-peaked preferences

Questions

- How peculiar is this example? I.e. when is the Pareto frontier convex?
- What is the meaning of fairness for random outcomes?
- Can fairness and efficiency be achieved?

Symmetry – an example

- $a \succ_1 b$ and $b \succ_2 a$
- The labels a and b don't have meaning
- Could rename $a \to b$ and $b \to a$
- Now the preference is $b \succ_1 a$ and $a \succ_2 b$
- The names of the agents also don't have any meaning
- Could rename $1 \to 2$ and $2 \to 1$ to get back $a \succ_1 b$ and $b \succ_2 a$
- 1 and 2 only differ in the labels we assign to agents and objects

Symmetry – another example

$$a \succ_1 b \succ_1 c$$
$$b \succ_2 c \succ_2 a$$
$$c \succ_3 a \succ_3 b$$

Permutation:

$$\sigma = \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}$$

Then:

$$b \succ_1^\sigma c \succ_1^\sigma a$$
$$c \succ_2^\sigma a \succ_2^\sigma b$$
$$a \succ_3^\sigma b \succ_3^\sigma c$$

Agent 1 is to this economy what 2 was to the first economy.

Symmetries of preference profiles

Given $\succsim_i$ on X and a permutation $\sigma : X \to X$, can form $\succsim_i^\sigma$ s.t.

$$x \succsim_i y \iff \sigma(x) \succsim_i^\sigma \sigma(y).$$

Consider the group $\Sigma_n \times \Sigma_m$ of pairs of permutations (σ, ν), with σ over X and ν over N.

Think of (σ, ν) as operating on preference profiles: $(\succsim_i)_{i \in N} \mapsto (\succsim_{\nu(i)}^\sigma)_{i \in N}$

For a profile $\succsim = (\succsim_i)_{i \in N}$, let $Z \subseteq \Sigma_n \times \Sigma_m$ be the subgroup that leaves $\succsim$ fixed: the *stabilizer subgroup*.

For the previous example

$$a \succ_1 b \succ_1 c$$
$$b \succ_2 c \succ_2 a$$
$$c \succ_3 a \succ_3 b$$

$$\sigma = \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix} \nu = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$
$$Z = \{(\sigma, \nu), \mathrm{Id}, (\sigma^{-1}, \nu^{-1})\}$$

Symmetries of preference profiles

A *symmetry* of a preference profile $(\succsim_i)_{i\in N}$ is a pair $(\sigma, \nu) \in Z$.
So

$$(\succsim^{\sigma}_{\nu(1)}, \succsim^{\sigma}_{\nu(2)}, \ldots, \succsim^{\sigma}_{\nu(n)}) = (\succsim_i)_{i\in N}$$

Agents i and j are *symmetric* if there is a symmetry (σ, ν) s.t. $\nu(i) = j$

Obs in that case, (σ^{-1}, ν^{-1}) is also a symmetry and $i = \nu^{-1}(j)$. So notion of symmetry is indeed symmetric.

Symmetries of preference profiles

We also have (σ, ν) operating on lotteries: $q \mapsto q^\sigma$.

The *orbit* of a lottery q is the set $O(q)$ obtained by the repeated application of permutations in Z.

Since Z is a group, this is the same as

$$O(q) = \{q^\sigma : (\sigma, \nu) \in Z\}.$$

A new notion of fairness

A lottery p satisfies *equal treatment of symmetric agents(ETOSA)* if for any symmetric agents i and j,

$$\sum_{x \in r_i(k)} p_x = \sum_{y \in r_j(k)} p_y$$

for all k.

Obs: the uniform lottery on X satisfies ETOSA, as $|r_i(k)| = |r_j(k)|$ when i and j are symmetric.

ETOSA

- Well defined for any social choice environment: Arrovian, house allocation, etc.
- A fixed-profile notion.
- Doesn't require a mechanism, or counterfactual choices.
- In house allocation, strictly stronger than equal treatment of equals

A new notion of fairness – back to the example

$$a \succ_1 x \succ_1 b \succ_1 y \succ_1 c \succ_1 z$$
$$b \succ_2 z \succ_2 c \succ_2 x \succ_2 a \succ_2 y$$
$$c \succ_3 y \succ_3 a \succ_3 z \succ_3 b \succ_3 x$$

- 1 and 2 are symmetric under the symmetry (σ, ν) where

$$\sigma = \begin{pmatrix} a & x & b & y & c & z \\ b & z & c & x & a & y \end{pmatrix} \text{ and } \nu = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

A new notion of fairness – back to the example

- All agents are symmetric $\implies p_a = p_b = p_c$ and $p_x = p_y = p_z$.
- Recall the uniform lottery $\frac{1}{6}a, \frac{1}{6}x, \frac{1}{6}b, \frac{1}{6}y, \frac{1}{6}c$ and $\frac{1}{6}z$
- And the alternative $\frac{1}{3}a, \frac{1}{3}b$ and $\frac{1}{3}c$
- Both satisfy ETOSA, but one is ex-ante efficient and the other is not.

Questions

- How peculiar is the example? I.e. when is the Pareto frontier convex?
- What is the meaning of fairness for random outcomes?
- Can fairness and efficiency be achieved?

How to improve on inefficient but fair outcomes

Proposition

Let p be any lottery that satisfies ETOSA. There is an ex-ante efficient lottery which dominates p and satisfies ETOSA.

How to improve on inefficient but fair outcomes

Notation:

$$\sum_{x \in A} p(x) = p(A)$$

Let p satisfy ETOSA but be inefficient.

Let q Pareto dominate p.

Fix (σ, ν) a symmetry of $(\succsim_i)_{i \in N}$ and $i = \nu(j)$.

Then $x \in r_i(k)$ iff $\sigma(x) \in r_j(k)$.

How to improve on inefficient but fair outcomes

Define q^σ by $q^\sigma(\sigma(x)) = q(x)$.

Then

$$q(A) = q^\sigma(\{\sigma(x) : x \in A\})$$
$$\implies q(r_i(k)) = q^\sigma(\{\sigma(x) : x \in r_i(k)\}) = q^\sigma(r_j(k))$$

So:

$$\underbrace{p(r_j(k)) = p(r_i(k))}_{\text{ETOSA}} \underbrace{\leq}_{q \text{ dom. } p} q(r_i(k)) = q^\sigma(r_j(k))$$

And thus q^σ also dominates p.

How to improve on inefficient but fair outcomes

Now consider the *orbit*

$$O(q) = \{q^\sigma : (\sigma, \nu) \in Z\}.$$

We have seen that if p is inefficient but fair, then for any dominating q and $(\sigma, \nu) \in Z$, q^σ also dominates p.

So all the elements of $O(q)$ dominate p. Let

$$\bar{q} = \frac{1}{|O(q)|} \sum_{\hat{q} \in O(q)} \hat{q}$$

be the average of $O(q)$.

$\bar{q}$ is a "symmetrization" of q.

How to improve on inefficient but fair outcomes

Claim: $\bar{q}$ is ETOSA and dominates p.

For any VNM utility u_i for $\succsim_i$, $u_i \cdot p \leq u_i \cdot q^\sigma$. Thus $u_i \cdot p \leq u_i \cdot \bar{q}$.

Moreover, if i and j are symmetric, then $\succsim_i = \succsim_j^\sigma$, and we have seen that $\hat{q}(r_i(k)) = \hat{q}^\sigma(r_j(k))$, for $(\sigma, \nu) \in Z$.

In consequence

$$\bar{q}(r_i(k)) = \frac{1}{|O(q)|} \sum_{\hat{q} \in O(q)} \hat{q}(r_i(k)) = \frac{1}{|O(q)|} \sum_{\hat{q} \in O(q)} \hat{q}^\sigma(r_j(k)) = \bar{q}(r_j(k))$$

as $O(q) = \{\hat{q}^\sigma : \hat{q} \in O(q)\}$ ($\hat{q} \mapsto \hat{q}^\sigma$ is a bijection).

How to improve inefficient outcomes

So we've found a lottery $\bar{q}$ that Pareto dominates p and satisfies ETOSA.

If $\bar{q}$ is not efficient, rinse and repeat.

Iterating, we should find an efficient lottery that satisfies ETOSA and dominates p.

Proof

Let P be the set of lotteries that weakly Pareto dominate p and satisfy ETOSA.

Obs: P is a polytope. It is defined by a finite set of linear inequalities:

1. $q(r_i(k)) \geq p(r_i(k))$ for all $i \in N$ and $1 \leq k \leq |X|$.
2. $q(r_i(k)) = q(r_i(k))$ for all symmetric i and j and $1 \leq k \leq |X|$.
3. $q \in \Delta(X)$.

And P is nonempty as it contains $\bar{q}$.

Proof

Let $q^* \in \text{argmax}\{\sum_i u_i(q) : q \in P\}$ for some choice of VNM u_i for $\succsim_i$.

Since $q^* \in P$ it satisfies ETOSA and dominates p. Indeed by def. q^* is not dominated by any lottery in P.

We claim that q^* is efficient.

For if is it not, then it is st. dominated by a lottery q^{**}. Starting from q^{**} we can again compute the orbit and take the mean, yielding a lottery that satisfies ETOSA and thus dominates q^* in P. This is absurd.