Discrete economies

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LAMES

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Two papers with Sumit and SangMok

- “Stable allocations in discrete economies”
- “Intertemporal assignments”
Primitives:

- A finite set of *agents* \( A \).
- A finite set of *objects* \( O \).
- For each agent \( i \in A \), a utility function \( v_i : 2^O \rightarrow \mathbb{R} \cup \{-\infty\} \).

An *allocation* is a disjoint collection \( \{X_i : i \in A\} \) with \( \bigcup_i X_i \subseteq O \).
Two broad questions (one per paper) regarding stability and fairness:

▸ In a model with endowments, what can we say about stable outcomes? (A generalization of TTC in the unit-demand housing market?)

▸ When objects are available over time, but there is unit demand in each period, what can we say about fair outcomes?
Intertemporal assignment.
Stanford Housing Office assigns undergraduate students a dorm on campus.

Basic idea: students rank housing options and their choices are executed based on a lottery (RSD).

Students are given two years in which they can draw a lottery number from a “preferred” urn with numbers 1-2000, and two years from an “unpreferred” urn with numbers 2001-4000.

The upshot is that students’ luck is negatively correlated over time: a basic fairness consideration.
So the last shall be first, and the first last: for many be called, but few chosen.

Matthew 20:16
Negative correlation in dynamic assignment

Negative correlation is not easy to square with the usual *ex-ante* view that motivates randomization and fairness.

Consider two children, Alice and Bob. Two periods: one piece of candy available each period. Fairness means that we flip a coin, but that’s only *ex-ante* fair.

If A and B are standard expected-utility, exponential discounting, agents:

\[
E_{u_i}(\tilde{X}_i^1) + \delta E_{u_i}(\tilde{X}_i^2)
\]

Fairness motivates randomization, but *ex-post* realizations are unfair. *Ex-ante*, intertemporal correlation are irrelevant.
Why negative correlation?

Our paper:

▶ Ex-post fairness.
▶ Intertemporal non-separability in social welfare.
▶ Incentives.

This talk: ex-post fairness.
An intertemporal model

$T$ time periods (in applications, $T$ small/moderate)

A finite set $O_t$ of objects; $O_t$ contains a null object $\emptyset$

An agent obtains an element of $O^T := \times_{t=1}^{T} O_t$.

So tuples $o = (o_1, \ldots, o_T)$. We may write $o$ as $(o_t, o_{-t})$.

Endow each agent $i$ with a preference $\succeq_i$ over $O^T$. 

Suppose that preferences are separable: there is a strict preference ordering $\succeq'_i$ over $\bigcup_{t=1}^{T} O_t$ so that $(o_t, o_{-t}) \succeq (o'_t, o_{-t})$ iff $o_t \succeq'_i o'_t$.

So if $o_t \succeq'_i o'_t$ for all $t$ then $o \succeq'_i o'$. Abusing notation denote $\succeq'_i$ by $\succeq_i$. 

An allocation is a function $\mu : A \rightarrow O^T$ s.t $\mu_t(i) \neq \emptyset$ means that $\mu_t(j) \neq \mu_t(i)$.

An allocation $\mu$ is *envy-free up to one good (EF1)* if there exists, for each $i$ and $j$, a time period $t$ such that

$$\mu(i) \succeq_i (\emptyset_t, \mu_{-t}(j)).$$

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1Lipton et al. (2004); Budish (2011)
An allocation $\mu$ is \textit{envy-free up to one exchange (EF1X)} if whenever $\mu(j) \succ_i \mu(i)$, there is $t$ s.t

$$(\mu_t(j), \mu_{-t}(i)) \succeq_i (\mu_t(i), \mu_{-t}(j)).$$

EF1X and EF1 are independent properties.

EF1 is less interesting when agents hate the null object. EF1X is less interesting when different objects have widely different cardinal values.

So we’d ideally want both!
Natural “last shall be first” mechanisms

- SD: Fix ordering of $A$. In each period, agents pick an object in order.
- Alternating SD: Fix an ordering $\succeq$ of the agents. In the first period, agents pick an object in the order given by $\succeq$. In the next period, they pick objects in the opposite order to $\succeq$. Then they alternate orders in each period.
- FIFO: Fix ordering $\succeq_1$ of $A$. Given an ordering $\succeq_t$, let $\succeq_{t+1}$ be obtained by moving the first agent in $\succeq_t$ to the last position, keeping the rest the same.

(+ random versions of each of these mechanisms)

**Proposition**

When $T = 2$, alt. SD is EF1 and EF1X. This is not true for $T > 2$. When $T \leq 4$ and preferences are additively separable, alt. SD is EF1X; not true for $T > 4$. FIFO may violate EF1 and EF1X.
$O = \{a, b, c\}, \ A = \{1, 2, 3\}, \ T > 2$

Preferences are the same for all agents and additively separable, $\sum_t v(o_t)$.

Let $v(a) = 100, \ v(b) = 2, \ v(c) = 1$ and $v(\emptyset) = 0$.

The order is $1 < 2 < 3$.

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<th>100</th>
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<tbody>
<tr>
<td>100</td>
<td>1</td>
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<td>2</td>
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<td>2</td>
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<tr>
<td>1</td>
<td>100</td>
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</tbody>
</table>

Then agent 2 will envy 1 even if set 1’s consumption in any period to $\emptyset$. 
Our mechanism

Let $\mu_{<t}$ denote an allocation of $O_1, O_2, \ldots, O_{t-1}$.

Assume each agent’s preference $\succeq_i$ over the bundles $\{\mu_{<t}(j) | j = 1, \ldots, n\}$ is strict.

Let $\mu_{ttc}$ be the outcome of TTC.

TTC partitions agents into groups $A_1, A_2, \ldots$ s.t agents in group $A_k$ were involved in the $k$-th cycle.

Choose an allocation $\mu_t$ of $O_t$ by SD, where agents in $A_k$ come before $A_{k-1}$ (otherwise arbitrary).
Our mechanism

Proposition

If $\mu_{<t}$ is EF1 (EF1X), then $\mu_{\leq t} = (\mu_{<t}^{ttc}, \mu_{t})$ is EF1 (EF1X).
Stable outcomes

Classical problem of exchange.

Textbook model of an exchange economy.

But with indivisible goods.

Most positive results are for either the housing (unit demand) model, or for large economies where “asymptotic convexity” comes to the rescue.
An **economy** is a tuple $E = (O, \{(v_i, \omega_i) : i \in A\})$ in which

- $O$ is a finite set of *objects*;
- $A$ is a finite set of *agents*;
- each agent $i \in A$ is described by
  - a utility function $v_i : 2^O \rightarrow \mathbb{R} \cup \{-\infty\}$ and a
  - nonempty endowment $\omega_i \subseteq O$, with $O = \bigcup_i \omega_i$ and $\omega_i \cap \omega_j = \emptyset$ when $i \neq j$.

An **$S$-allocation** is a disjoint collection of sets of objects $\{X_i : i \in S\}$ such that $\bigcup_{i \in S} X_i \subseteq \bigcup_{i \in S} \omega_i$.

An allocation $X$ is in the **weak core** if there is no coalition $S \subseteq A$ and $S$-allocation $Y$ such that $v_i(Y_i) > v_i(X_i)$ for all $i \in S$. 
Consider economy $E$ such that $|\omega_i| = 1$ and
\[ v_i(X) = -\infty \iff |X| \neq 1. \]
TTC algorithm leads to an allocation in the core.
Question: What can we say about the core in more general economies?
Our paper

- Sufficient conditions for nonemptiness of the core.
- An algorithm that generalizes TTC (based on Adachi’s $T$-algorithm).
Additive utilities

- Suppose $v_i(X) = \sum_{x \in X} u_i(x)$
- Konishi, Quint, and Wako (2001) give an example with $|A| = 4, |O| = 5$ and additive utilities in which the weak core is empty.
- What if we further restrict utilities to be dichotomous so that $u_i(x) \in \{0, 1\}$?
Theorem

Suppose $E$ is such that $v_i(X) = \sum_{x \in X} u_i(x)$ where $u_i(x) \in \{0, 1\}$. Then the weak core is non-empty.

- Given economy $E$, the resulting NTU game is given by $(A, V)$ where

$$V(S) = \{u \in \mathbb{R}^A : \exists S - \text{ allocation } X \text{ s.t. } u_i \leq v_i(X_i) \text{ for all } i \in S\}$$

- Scarf (1967)'s condition: For any balanced collection of coalitions $S$ ($\sum_{\{S \in S : \exists i\}} \delta_S = 1$ for all $i \in A$):

$$u \in \bigcap_{S \in S} V(S) \implies u \in V(A)$$

- Let

$$P = \sum_{S \in S} \delta_S P_S$$

Observe that $P$ has two properties:

1. Row $i$ sums up to at least $u_i \geq 1$.
2. Each column sums up to at most 1.
The existence result generalizes when there are categories of objects.

An economy $E$ is **categorical** if there exists $K$ categories, and sets of objects $O^k$ for each category $k \in \{1 \ldots K\}$; so that

- agents may consume at most one object of each category, and
- agents’ utility is dichotomous in each category, and
- additively separable over categories.

See Moulin (2014).

**Theorem**

A categorical economy has a nonempty weak core.
The following is an example of a categorical economy with non-additive utilities in which the weak core is empty.

Example

$A = \{1, 2, 3\}$, endowment $\omega_i = (l_i, r_i)_{i=1,2,3}$ and utilities representing the following preferences:

<table>
<thead>
<tr>
<th>Agent 1</th>
<th>Agent 2</th>
<th>Agent 3</th>
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<tbody>
<tr>
<td>$(l_1, r_2)^1$</td>
<td>$(l_2, r_3)^2$</td>
<td>$(l_1, r_3)^3$</td>
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<tr>
<td>$(l_3, r_1)^3$</td>
<td>$(l_2, r_1)^1$</td>
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Existence 3: Gains from trade

- An economy satisfies \textit{gains from trade} if, for any two coalitions \( S \) and \( S' \), any \( S \)-allocation \( X \) and any \( S' \)-allocation \( X' \), there exists an \( S \cup S' \)-allocation \( Y \) with

\[
\nu_h(Y_h) \geq \min\{ \nu_h(X_h), \nu_h(X'_h) \}
\]

for all \( h \in S \cup S' \); where we define \( X'_h = X_h \) when \( h \in S' \setminus S \) or \( h \in S' \setminus S \).

\textbf{Theorem}

If an economy has gains from trade, and all agents’ utilities are strict, then the weak core is nonempty.

- The resulting NTU game is ordinally convex

\[
V(S) \cap V(S') \subseteq V(S \cap S') \cup V(S \cup S').
\]

and therefore has a non-empty core.
Suppose $E = (O, \{(v_i, \omega_i) : i \in A\})$ is such that $v_i$ is strict and monotone increasing.

Let $U_i = v_i(2^O)$, $U = \times U_i$ and define $T_k : U \to U$ by $$(T_k u)_i = \max B_i^k(u)$$ where

$$B_i^k(u) = \{ u'_i : \exists S\text{-allocation } X \text{ st } |S| \leq k, i \in S, v_i(X_i) = u'_i \text{ and for all } j \in S \setminus \{i\}; v_j(X_j) \geq u_j \}.$$ 

The $m$th iterate of $T_k$ is defined recursively as $T_k^m u = T_k(T_k^{m-1} u)$.
Given a housing market $E$, the TTC defines a partition of agents, $A_1, A_2, \ldots, A_k$, so that agents in $A_r$ get their final allocation in round $r$ of the algorithm. Let $\bar{u}$ denote the TTC utilities.

Let $u_i = v_i(\omega_i)$ and consider repeated application of $T$ starting from $u$.

**Lemma**

If agent $i \in A_r$, $(T^m(u))_i = \bar{u}_i$ for $m \geq 2r - 1$. 
Lemma

There exists $u \in U$ such that $u = T^2 u$, $u \leq Tu$.

If $u = Tu$, then $\{v_i^{-1}(u_i) : i \in A\}$, is an allocation that is individually rational and not blocked by any coalition of size at most $k$. 
Related literature

TTC: Roth and Postlewaite (1977), Ma (1994), Sönmez (1999), Sönmez and Ünver (2010), and Roth et al. (2004)
