Discrete economies

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LAMES

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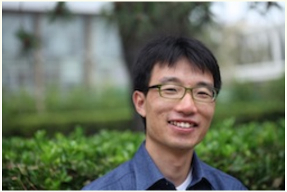
Two papers with Sumit and SangMok



- "Stable allocations in discrete economies"
- "Intertemporal assignments"

Discrete allocation problem

Primitives:

- A finite set of *agents* A.
- A finite set of *objects* O
- For each agent $i \in A$, a utility function $v_i : 2^O \to \mathrm{R} \cup \{-\infty\}$.

An *allocation* is a disjoint collection $\{X_i : i \in A\}$ with $\cup_i X_i \subseteq O$.

Discrete allocation problem

Two broad questions (one per paper) regarding stability and fairness:

- In a model with endowments, what can we say about stable outcomes? (A generalization of TTC in the unit-demand housing market?)
- When objects are available over time, but there is unit demand in each period, what can we say about fair outcomes?

Intertemporal assignment.

Negative correlation in The Farm

Stanford Housing Office assigns undergraduate students a dorm on campus.

Basic idea: students rank housing options and their choices are executed based on a lottery (RSD).

Students are given two years in which they can draw a lottery number from a "preferred" urn with numbers 1-2000, and two years from an "unpreferred" urn with numbers 2001-4000.

The upshot is that students' luck is negatively correlated over time: a basic fairness consideration.

Negative correlation in the bible

So the last shall be first, and the first last: for many be called, but few chosen.

Matthew 20:16

Negative correlation in dynamic assignment

Negative correlation is not easy to square with the usual *ex-ante* view that motivates randomization and fairness.

Consider two children, Alice and Bob. Two periods: one piece of candy available each period. Fairness means that we flip a coin, but that's only ex-ante fair.

If A and B are standard expected-utility, exponential discounting, agents:

$$\mathsf{E}u_i(\tilde{X}_i^1) + \delta \mathsf{E}u_i(\tilde{X}_i^2)$$

Fairness motivates randomization, but ex-post realizations are unfair.
Ex-ante, intertemporal correlation are irrelevant.

Why negative correlation?

Our paper:

- Ex-post fairness.
- Intertemporal non-separability in social welfare.
- Incentives.

This talk: ex-post fairness.

An intertemporal model

T time periods (in applications, T small/moderate)

A finite set O_t of objects; O_t contains a null object $\emptyset$

An agent obtains an element of $O^T := \times_{t=1}^{T} O_t$.

So tuples $o = (o_1, \ldots, o_T)$. We may write o as (o_t, o_{-t}).

Endow each agent i with a preference $\succeq_i$ over O^T.

An intertemporal model

Suppose that preferences are separable: there is a strict preference ordering $\succeq_i'$ over $\cup_{t=1}^{T} O_t$ so that $(o_t, o_{-t}) \succeq (o_t', o_{-t})$ iff $o_t \succeq_i' o_t'$.

So if $o_t \succeq_i' o_t'$ for all t then $o \succeq_i o'$. Abusing notation denote $\succeq_i'$ by $\succeq_i$.

Ex-post fairness

An allocation is a function $\mu : A \to O^T$ s.t $\mu_t(i) \neq \emptyset$ means that $\mu_t(j) \neq \mu_t(i)$.

An allocation μ is *envy-free up to one good (EF1)* if there exists, for each i and j, a time period t such that

$$\mu(i) \succeq_i (\emptyset_t, \mu_{-t}(j)).^1$$

[1] Lipton et al. (2004); Budish (2011)

Ex-post fairness

An allocation μ is *envy-free up to one exchange (EF1X)* if whenever $\mu(j) \succ_i \mu(i)$, there is t s.t

$$(\mu_t(j), \mu_{-t}(i)) \succeq_i (\mu_t(i), \mu_{-t}(j)).$$

EF1X and EF1 are independent properties.

EF1 is less interesting when agents hate the null object. EF1X is less interesting when different objects have widely different cardinal values.

So we'd ideally want both!

Natural "last shall be first" mechanisms

- SD: Fix ordering of A. In each period, agents pick an object in order
- Alternating SD: Fix an ordering $\geq$ of the agents. In the first period, agents pick an object in the order given by $\geq$. In the next period, they pick objects in the opposite order to $\geq$. Then they alternate orders in each period.
- FIFO: Fix ordering $\geq_1$ of A. Given an ordering $\geq_t$, let $\geq_{t+1}$ be obtained by moving the first agent in $\geq_t$ to the last position, keeping the rest the same.

($+$ random versions of each of these mechanisms)

Proposition

When $T = 2$, alt. SD is EF1 and EF1X. This is not true for $T > 2$. When $T \leq 4$ and preferences are additively separable, alt. SD is EF1X; not true for $T > 4$. FIFO may violate EF1 and EF1X.

Ex with alt. SD

$O = \{a, b, c\}$, $A = \{1, 2, 3\}$, $T > 2$

Preferences are the same for all agents and additively separable, $\sum_t v(o_t)$.

Let $v(a) = 100$, $v(b) = 2, v(c) = 1$ and $v(\emptyset) = 0$.

The order is $1 < 2 < 3$.

100	1	100
2	2	2
1	100	1

Then agent 2 will envy 1 even if set 1's consumption in any period to $\emptyset$.

Our mechanism

Let $\mu_{<t}$ denote an allocation of $O_1, O_2, \ldots, O_{t-1}$.

Assume each agent's preference $\succeq_i$ over the bundles $\{\mu_{<t}(j) | j = 1, \ldots, n\}$ is strict.

Let $\mu^{ttc}_{<t}$ be the outcome of TTC.

TTC partitions agents into groups $A_1, A_2, \ldots$ s.t agents in group A_k were involved in the k-th cycle.

Choose an allocation μ_t of O_t by SD, where agents in A_k come before A_{k-1} (o/wise arbitrary).

Our mechanism

Proposition

If $\mu_{<t}$ is EF1 (EF1X), then $\mu_{\leq t} = (\mu_{<t}^{ttc}, \mu_t)$ is EF1 (EF1X).

Stable outcomes

Classical problem of exchange.

Textbook model of an exchange economy.

But with indivisible goods.

Most positive results are for either the housing (unit demand) model, or for large economies where "asymptotic convexity" comes to the rescue.

Model

- An *economy* is a tuple $E = (O, \{(v_i, \omega_i) : i \in A\})$ in which
  - O is a finite set of *objects*;
  - A is a finite set of *agents*;
  - each agent $i \in A$ is described by
    - a utility function $v_i : 2^O \to \mathrm{R} \cup \{-\infty\}$ and a
    - nonempty endowment $\omega_i \subseteq O$, with $O = \cup_i \omega_i$ and $\omega_i \cap \omega_j = \emptyset$ when $i \neq j$.
- An *S-allocation* is a disjoint collection of sets of objects $\{X_i : i \in S\}$ such that $\cup_{i \in S} X_i \subseteq \cup_{i \in S} \omega_i$.
- An allocation X is in the *weak core* if there is no coalition $S \subset A$ and $S - allocation\ Y$ such that $v_i(Y_i) > v_i(X_i)$ for all $i \in S$.

Housing market (Shapley and Scarf (1974))

- Consider economy E such that $|\omega_i| = 1$ and $v_i(X) = -\infty \iff |X| \neq 1$.
- TTC algorithm leads to an allocation in the core.
- Question: What can we say about the core in more general economies?

Our paper

- Sufficient conditions for nonemptyness of the core.
- An algorithm that generalizes TTC (based on Adachi's T-algorithm).
- A bargaining set/stable set solution. Reminiscent of Roth (1975).

Additive utilities

- Suppose $v_i(X) = \sum_{x \in X} u_i(x)$
- Konishi, Quint, and Wako (2001) give an example with $|A| = 4, |O| = 5$ and additive utilities in which the weak core is empty.
- What if we further restrict utilities to be dichotomous so that $u_i(x) \in \{0, 1\}$?.

Existence 1

Theorem

Suppose E is such that $v_i(X) = \sum_{x \in X} u_i(x)$ where $u_i(x) \in \{0, 1\}$. Then the weak core is non-empty.

- Given economy E, the resulting NTU game is given by (A, V) where

$$V(S) = \{u \in \mathbb{R}^A : \exists S - \text{allocation X s.t. } u_i \leq v_i(X_i) \text{ for all } i \in S\}$$

- Scarf (1967)'s condition: For any balanced collection of coalitions $\mathcal{S}$ ($\sum_{\{S \in \mathcal{S}: S \ni i\}} \delta_S = 1$ for all $i \in A$):

$$u \in \cap_{S \in \mathcal{S}} V(S) \implies u \in V(A)$$

- Let

$$P = \sum_{S \in \mathcal{S}} \delta_S P_S$$

Observe that P has two properties:

1. Row i sums up to at least $u_i \geq 1$.
2. Each column sums up to at most 1.

Existence 2: Categorical economies

- The existence result generalizes when there are categories of objects.
- An economy E is *categorical* if there exists K categories, and sets of objects O^k for each category $k \in \{1 \dots K\}$; so that
  - agents may consume at most one object of each category, and
  - agents' utility is dichotomous in each category, and
  - additively separable over categories.
  - See Moulin (2014).

Theorem

A categorical economy has a nonempty weak core.

Examples

The following is an example of a categorical economy with non-additive utilities in which the weak core is empty.

Example

$A = \{1, 2, 3\}$, endowment $\omega_i = (l_i, r_i)_{i=1,2,3}$ and utilities representing the following preferences:

Agent 1	Agent 2	Agent 3
$(l_1, r_2)^1$	$(l_2, r_3)^2$	$(l_1, r_3)^3$
$(l_3, r_1)^3$	$(l_2, r_1)^1$	$(l_3, r_2)^2$
$(l_1, r_1)^2$	$(l_2, r_2)^3$	$(l_3, r_3)^1$
$\dots$	$\dots$	$\dots$

Existence 3: Gains from trade

- An economy satisfies *gains from trade* if, for any two coalitions S and S', any S-allocation X and any S'-allocation X', there exists an $S \cup S'$-allocation Y with

$$v_h(Y_h) \geq \min\{v_h(X_h), v_h(X'_h)\}$$

for all $h \in S \cup S'$; where we define $X'_h = X_h$ when $h \in S' \setminus S$ or $h \in S' \setminus S$.

Theorem

If an economy has gains from trade, and all agents' utilities are strict, then the weak core is nonempty.

- The resulting NTU game is ordinally convex

$$V(S) \cap V(S') \subseteq V(S \cap S') \cup V(S \cup S').$$

and therefore has a non-empty core.

An algorithm

- Suppose $E = (O, \{(v_i, \omega_i) : i \in A\})$ is such that v_i is strict and monotone increasing.
- Let $U_i = v_i(2^O)$, $U = \times U_i$ and define $T_k : U \to U$ by $(T_k u)_i = \max B_i^k(u)$ where

$$B_i^k(u) = \{ u_i' : \exists S\text{-allocation } X \text{ st } |S| \leq k, i \in S, v_i(X_i) = u_i' \\ \text{and for all } j \in S \setminus \{i\}; v_j(X_j) \geq u_j \} .$$

- The mth iterate of T_k is defined recursively as $T_k^m u = T_k(T_k^{m-1} u)$

Generalization of TTC

- Given a housing market E, the TTC defines a partition of agents, $A_1, A_2, \ldots, A_k$, so that agents in A_r get their final allocation in round r of the algorithm. Let $\bar{u}$ denote the TTC utilities.
- Let $\underline{u}_i = v_i(\omega_i)$ and consider repeated application of T starting from $\underline{u}$.

Lemma

If agent $i \in A_r$, $(T^m(\underline{u}))_i = \bar{u}_i$ for $m \geq 2r - 1$.

k-stable

Lemma

- There exists $u \in U$ such that $u = T^2 u$, $u \leq Tu$.
- If $u = Tu$, then $\{v_i^{-1}(u_i) : i \in A\}$, is an allocation that is individually rational and not blocked by any coalition of size at most k.

Related literature

TTC: Roth and Postlewaite (1977), Ma (1994), Sönmez (1999), Sönmez and Ünver (2010), and Roth et al. (2004)

General problem of exchange: Henry (1970) Mas-Colell (1977), Quinzii (1984), Svensson (1983), Inoue (2008), Budish (2011), Baldwin, Edhan, Jagadeesan, Klemperer, and Teytelboym (2020), Jagadeesan and Teytelboym (2021)

T-algorithm: Roth (1975), Adachi (2000), Echenique and Oviedo (2004), Echenique and Oviedo (2006), Fleiner (2003), Hatfield and Milgrom (2005), Echenique and Yenmez (2007), and Ostrovsky (2008).