Stable matching and optimal transport

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Matching with aligned preferences

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Ferdowsian, Niederle and Yariv (2020)
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Ferdowsian, Niederle and Yariv (2020)
Agents are described by their “position” on the real line.

Two populations: \( \mu \) on \( X \subseteq \mathbb{R} \) and \( \nu \) on \( Y \subseteq \mathbb{R} \), each with a density.

If \( x \in X \) and \( y \in Y \) match, each gets utility

\[
 u(x, y) = -|x - y|^2.
\]
An example

Populations:

$X$ and $Y$
An example
An example
An example

Structural property of stable matchings: no crossing.
Another example: stability vs. fairness
Emerge naturally:

When there’s an objective “fit,” or match-quality.

When there are transfers, but these can’t be bargained over at the matching stage.
This paper: Aligned preferences are interesting!

Connection to optimal transport.

Tension between stability and fairness.

Approximate utilitarian welfare max.

Structural properties of stable matchings.

General matching model, encompassing finite and infinite markets.

Results extend to many-sided matching, where existence is not generally guaranteed.

For ex. team formation, supply chains, organ donation . . .
A two-sided matching problem is a tuple \(((X, \mu), (Y, \nu), u, v)\):

- \(X\) and \(Y\) are Polish spaces endowed with their Borel \(\sigma\)-algebra.
- \(\mu\) is a finite measure on \(X\).
- \(\nu\) is a finite measure on \(Y\).
- \(\mu(X) = \nu(Y)\).
- \(u: X \times Y \to \mathbb{R}\) and \(v: X \times Y \to \mathbb{R}\) are measurable.

Interpretation: two populations, \((X, \mu)\) and \((Y, \nu)\).
The utility of \(x \in X\) from matching with \(y \in Y\) is \(u(x, y)\) and the utility for \(y\) is \(v(x, y)\).
Let $\mathcal{M}_+(X \times Y)$ be the space of all finite measures on $X \times Y$.

$\pi \in \mathcal{M}_+(X \times Y)$ is a matching if it has marginal $\mu$ on $X$ and $\nu$ on $Y$. Denote by $\Pi(\mu, \nu)$ the set of all matchings.
Criteria for matchings

Stability

Egalitarianism

Utilitarian welfare
Stability

**Definition**

A matching $\pi$ is $\varepsilon$-stable, with $\varepsilon \geq 0$, if for $\pi \times \pi$-almost all pairs $(x_1, y_1)$ and $(x_2, y_2)$ at least one of these holds:

\[
\begin{align*}
    u(x_1, y_1) - u(x_1, y_2) & \geq -\varepsilon \\
    v(x_2, y_2) - v(x_1, y_2) & \geq -\varepsilon
\end{align*}
\]

If the inequalities are violated, $(x_1, y_2)$ is called an $\varepsilon$-blocking pair.
An equivalent notion of stability:

**Lemma**

Let $u$ and $v$ be continuous utility functions. 

$\pi$ is $\varepsilon$-stable iff for any $(x_1, y_1), (x_2, y_2) \in \text{supp}(\pi)$, at least one of these holds:

\[ u(x_1, y_1) - u(x_1, y_2) \geq -\varepsilon \]
\[ v(x_2, y_2) - v(x_1, y_2) \geq -\varepsilon \]
Aligned preferences

**Definition**
Preferences of agents in X and Y are aligned if \( u = v \).

Stability for markets with aligned preferences boils down to a single inequality.

A matching \( \pi \) is stable, if for \( \pi \times \pi \)-almost all pairs \((x_1, y_1)\) and \((x_2, y_2)\),

\[
 u(x_1, y_2) \leq \max \{ u(x_1, y_1), u(x_2, y_2) \} + \varepsilon.
\]
To each $\pi \in \Pi(\mu, \nu)$ define

$$U_{\min}(\pi) = \min_{(x, y) \in \text{supp}[\pi]} u(x, y).$$

When $X$ and $Y$ are cpct. and $u$ is cont. $U_{\min}(\pi)$ is well-defined.

Otherwise replace minimum with infimum.
Egalitarian matching

Definition

A matching $\pi$ is egalitarian if $U_{\text{min}}(\pi)$ is maximal among all matchings.

Definition

A matching is $(\epsilon, \delta)$-egalitarian if there is a subset $S \subset X \times Y$ with $\pi(S) \geq 1 - \delta$ s.t. $\forall (x, y) \in S$, $u(x, y)$ is at least $U_{\text{min}}(\pi^*) - \epsilon$, where $\pi^*$ is an egalitarian matching.
Define the utilitarian welfare of a matching \( \pi \) by

\[
W(\pi) = \int_{X \times Y} u(x, y) \, d\pi(x, y)
\]
Given a measurable *cost function* \( c : X \times Y \rightarrow \mathbb{R} \).

Goal is find a matching \( \pi \in \Pi(\mu, \nu) \) that minimizes the total cost:

\[
\min_{\pi \in \Pi(\mu, \nu)} \int_{X \times Y} c(x, y) \, d\pi(x, y),
\]
Let \(((X, \mu), (Y, \nu), u, u)\) be a two-sided matching market with aligned preferences.

Consider the following cost function

\[ c_\alpha(x, y) = 1 - \frac{\exp(\alpha \cdot u(x, y))}{\alpha}, \]

Obs. the different meanings of the optimal transport problem when \(\alpha > 0\) and when \(\alpha < 0\).
Theorem

Let $\pi^*$ be a solution to the optimal transport problem

$$
\min_{\pi \in \Pi(\mu, \nu)} \int_{X \times Y} c_\alpha(x, y) \, d\pi(x, y).
$$

1. If $\alpha > 0$ then $\pi^*$ is $\varepsilon$-stable, with $\varepsilon = (\ln 2)/\alpha$.
2. If $\alpha < 0$ then $\pi^*$ is $(\varepsilon, \delta)$-egalitarian, with $\varepsilon > 0$ and $0 < \delta \leq \exp(\alpha \cdot \varepsilon)$.
3. If $\alpha = 0$ then for any $\varepsilon$-stable matching $\pi$ we have

$$
W(\pi) \geq \frac{1}{2} (W(\pi^*) - \varepsilon).
$$
The result on stability is not specific to the exponential objective.

May assume:

\[ c(x, y) = -h(u(x, y)) . \]

If

\[ \frac{h'(t)}{h(t)} \geq \alpha \]

for \( t \) in the range of \( u \), then any solution to the transportation problem is \( \varepsilon \)-stable with \( \varepsilon = \frac{\ln 2}{\alpha} \).
Existence of stable matchings

Let $\Pi^u_\infty(\mu, \nu)$ the set of matchings that are the weak limit of solutions to the transportation problem with cost $c_\alpha$ for some sequence of $\alpha \to \infty$.

**Corollary**

For continuous $u$, let $\tilde{u}$ be a monotone reparametrization of $u$ s.t $\tilde{u}$ is bounded. Then $\Pi^{\tilde{u}}_\infty(\mu, \nu)$ is a non-empty convex weakly-closed set of stable matchings.

In particular, a stable matching exists for any market with aligned preferences and continuous $u$ and can be obtained as the limit of solutions to a sequence of optimal transport problems.
Suppose that $X$ and $Y$ are subsets of $\mathbb{R}$ and $\mu$ and $\nu$ are abs. cont. wrt Lebesgue measure.

Utility is

$$u(x, y) = -|x - y|.$$ 

Consider the problem:

$$\min_{\pi \in \Pi(\mu, \nu)} \int_{\mathbb{R} \times \mathbb{R}} f(|x - y|) \, d\pi(x, y),$$

(1)
The assortative matching is supported on the curve given by the equation $F_\mu(x) = F_\nu(y)$, where $F_\mu$ and $F_\nu$ are the cumulative distribution functions of $\mu$ and $\nu$, respectively.

This is a Monge plan with $m = F_\nu^{-1} \circ F_\mu$.

If $f$ is strictly concave, the assortative matching is the only optimal one.

Hence: $X = Y = \mathbb{R}$ and non-atomic $\mu, \nu$ with bounded support, the assortative matching is optimal for $c_\alpha$ for any $\alpha \leq 0$. For $\alpha < 0$, it is the unique optimal one.

**Corollary**

For $X = Y = \mathbb{R}$ and non-atomic $\mu, \nu$ with bounded support, the assortative matching is egalitarian and utilitarian welfare-maximizing.
A result due to McCann:
Let $O(x, y)$ be the smallest circle in $\mathbb{R}^2$ intersecting $(x, 0)$ and $(y, 0)$.

**Lemma**

For $X = Y = \mathbb{R}$, given non-atomic $\mu$ and $\nu$ let $\gamma$ be the optimal transportation plan when $f$ is strictly concave. Then for any $(x, y)$ and $(x', y')$ in the support of $\gamma$, $O(x, y)$ and $O(x', y')$ don’t intersect unless $x = x'$ or $y = y'$. 
Cyclic monotonicity

Definition

Given $c : X \times Y \to \mathbb{R}$, $\Gamma \subset X \times Y$ is called $c$-cyclic monotone if

$$\sum_{i=1}^{n} c(x_i, y_i) \leq \sum_{i=1}^{n} c(x_i, y_{i+1})$$

for all $n \geq 2$ and pairs of points $(x_1, y_1), \ldots, (x_n, y_n) \in \Gamma$ with the convention $y_{n+1} = y_1$.

Idea: if $\pi^*$ solves the optimal transport problem, then it is supported on a $c$-cyclic monotone set.
Proof of 1st statement

Use cyclic monotonicity.

For

\[ c(x, y) = -e^{\alpha u(x,y)}, \]

on the support of \( \pi^* \),

\[ c(x_1, y_1) + c(x_2, y_2) \leq c(x_1, y_2) + c(x_2, y_1) \]

Then

\[ e^{\alpha u(x_1,y_2)} + e^{\alpha u(x_2,y_1)} \leq e^{\alpha u(x_1,y_1)} + e^{\alpha u(x_2,y_2)} \]
Proof of 1st statement

\[ e^{\alpha u(x_1, y_2)} \leq \exp(\alpha u(x_1, y_1)) + \exp(\alpha u(x_2, y_2)) \]
\[ \leq 2 \max\{\exp(\alpha u(x_1, y_1)), \exp(\alpha u(x_2, y_2))\} \]
\[ = 2 \exp[\alpha \max\{u(x_1, y_1), u(x_2, y_2)\}] \]

Hence:

\[ \alpha u(x_1, y_2) \leq \ln(2) + \alpha \max\{u(x_1, y_1), u(x_2, y_2)\} \]

Since \( \frac{\ln 2}{\alpha} = \varepsilon \) we are done.
Denote $U_{\min}(\pi^*)$ by $U^*$ for short and let

$$C = \{(x, y) \in X \times Y : u(x, y) < U_{\min}(\pi^*) - \varepsilon\}.$$  

Note that when $\alpha < 0$ the solutions to the optimal transport problem are equal to the solutions of maximizing $\int -e^{\alpha \cdot u(x, y)} \, d\pi$, for $\pi \in \Pi(\mu, \nu)$. Let $\pi$ be a solution to this problem.
Proof of 2nd statement

Now observe that

\[-e^{\alpha \cdot U^*} \leq \int_{X \times Y} -e^{\alpha \cdot u(x,y)} \, d\pi^*(x, y) \]

\[\leq \int_{X \times Y} -e^{\alpha \cdot u(x,y)} \, d\pi(x, y) \]

\[= \int_{C} -e^{\alpha \cdot u(x,y)} \, d\pi(x, y) + \int_{X \times Y \setminus C} -e^{\alpha \cdot u(x,y)} \, d\pi(x, y) \]

\[\leq \int_{C} -e^{\alpha \cdot u(x,y)} \, d\pi(x, y) \]

\[\leq -e^{\alpha \cdot [U^* - \varepsilon]} \pi(C). \]

Thus \( \alpha U^* \geq \alpha (U^* - \varepsilon) + \ln(\pi(C)) \), so \( \alpha \varepsilon \geq \ln(\pi(C)) \).
Recall that \( \pi \in \Pi(\mu, \nu) \) is \( \varepsilon \)-stable with \( \varepsilon \geq 0 \) if for \( \pi \times \pi \)-almost all pairs \( (x_1, y_1) \) and \( (x_2, y_2) \)

\[
    u(x_1, y_2) \leq \max\{u(x_1, y_1), u(x_2, y_2)\} + \varepsilon.
\]

The couple \( (x_1, y_2) \) has distribution \( \mu \)

A matching \( \pi \) is \( \varepsilon \)-stable if, for a generic couple \( (x, y) \), the utility of at least one of the partners \( x \) or \( y \) in \( \pi \) is at least the utility of \( (x, y) \) minus \( \varepsilon \).
A weak order (a complete and transitive binary relation) is termed a \textit{preference}. If the weak order $\succeq$ is over a topological space $Z$, then we say that it is \textit{continuous} if the upper contour sets $U_{\succeq}(z) = \{z' \in Z : z' \succ z\}$ and lower contour sets $L_{\succeq}(z) = \{z' \in Z : z' \succ z\}$ are open.
Primitives are a tuple \((X, Y, \succeq_X, \succeq_Y)\) in which:

- \(X\) and \(Y\) are topological spaces.
- \(\succeq_X = \{\succeq_x : x \in X\}\), where for each \(x \in X\), \(\succeq_x\) is a continuous preference on \(Y\);
- \(\succeq_Y = \{\succeq_Y : y \in Y\}\) for each \(y \in Y\), \(\succeq_Y\) is a continuous preference on \(X\).
A function $u : X \times Y \to \mathbb{R}$ is a potential for $(X, Y, \succeq_X, \succeq_Y)$ if

- $u(x, y) \geq u(x, y')$ iff $y \succeq_x y'$
- and $u(x, y) \geq u(x', y)$ iff $x \succeq_y x'$.
1. If $b \succ_a b'$ then there is a neighborhood $N$ of $a$ for which $b \succ_{\tilde{a}} b'$ whenever $\tilde{a} \in N$.

2. If $b' \preceq_a b$ and $b \preceq_{a'} b''$ then, in any neighborhood of $b$, there exists $\hat{b}$ with $b' \succ_a \hat{b}$ and $\hat{b} \succ_{a'} b''$. 
The environment \((X, Y, \succeq_X, \succeq_Y)\) is *acyclic* if, for any sequence of couples,

\[(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n),\]

with \(n > 2\) and \((x_n, y_n) = (x_1, y_1)\), so that each couple \((x_{i+1}, y_{i+1})\) has exactly one agent in common with the preceding couple \((x_i, y_i)\), whenever all the common agents rank their partner in \((x_{i+1}, y_{i+1})\) above their partner in \((x_i, y_i)\), all common agents are in fact indifferent between the two partners.
Existence of a potential

Let \((X, Y, \succeq_X, \succeq_Y)\) be such that \(X\) and \(Y\) are complete, separable and connected topological spaces. Suppose that it satisfies acyclicity and properties (1) and (2). Then there is a potential for \((X, Y, \succeq_X, \succeq_Y)\).