What Matchings Can be Stable? Refutability in Matching Theory

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April 21-22, 2006

Motivation

Standard problem in matching theory.

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New problem — Given:

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Are there preferences s.t. $\mu_1, \ldots, \mu_K$ are stable ?
i.e. can you *rationalize* $\mu_1, \ldots, \mu_K$ using matching theory ?

Results – vaguely

Testing (Two-sided) Matching Theory:
Given

- Observations: agents & matchings (who matches to whom).
- Unobservables: preferences.

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Refutability in Economics

- Consumer and producer theory: Samuelson, Afriat, Varian, Diewert, McFadden, Hanoch & Rothschild, Richter, Matzkin & Richter.
- General Equilibrium Theory: Sonnenschein, Mantel, Debreu, Mas-Colell, Brown & Matzkin, Brown & Shannon, Kübler, Bossert & Sprumont, Chappori, Ekeland, Kübler & Polemarchakis.
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- Matching ?

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- Matching as a positive theory.
  many recent empirical papers on matching.
- Applications:
  - Marriages of "types."
  - Hospital-interns matches outside the NRMP.
  - Student-schools outside of NY.

The Model

Two finite, disjoint, sets M (men) and W (women).

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A *matching* is a function $\mu : M \cup W \rightarrow M \cup W \cup \{\emptyset\}$ s.t.

1. $\mu(w) \in M \cup \{\emptyset\}$,
2. $\mu(m) \in W \cup \{\emptyset\}$,
3. and $m = \mu(w)$ iff $w = \mu(m)$.

Denote the set of all matchings by $\mathcal{M}$.

The Model – Preferences

A *preference relation* is a linear, transitive and antisymmetric binary relation.

$P(m)$ is over $W \cup \{\emptyset\}$
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$$P = \left((P(m))_{m \in M}, (P(w))_{w \in W}\right).$$

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Note that preferences are strict.

Stability – Definition

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$$\mu(a) \neq \emptyset \Rightarrow \mu(a)\ P(a)\ \emptyset.$$

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$S(P)$ is the set of stable matchings.

Gale-Shapley (1962)

Theorem

$S(P)$ is non-empty and
$\exists$ a man-best/woman-worst and a woman-best/man-worst matching.

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Let $\mathcal{H} = \{\mu_1, \ldots \mu_n\} \subseteq \mathcal{M}$. Is there a preference profile P such that $\mathcal{H} \subseteq S(P)$?

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Let $\mathcal{H} = \{\mu_1, \dots \mu_n\} \subseteq \mathcal{M}$. Is there a preference profile P such that $\mathcal{H} \subseteq S(P)$?

Say that $\mathcal{H}$ can be *rationalized* if there is such P.

Let $|M| = |W|$.

$\mu(a) \neq \emptyset$ for all a and all $\mu \in \mathcal{H}$.

(this is WLOG)

Proposition

If $|M| \geq 3$, then $\mathcal{M}$ is not rationalizable.

Proof

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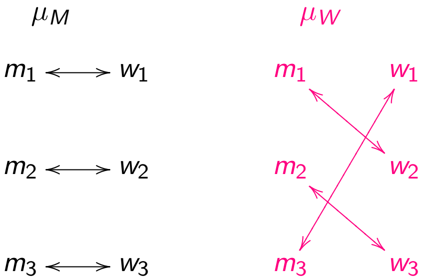
$$\mu_M$$

$$m_1 \longleftrightarrow w_1$$

$$m_2 \longleftrightarrow w_2$$

$$m_3 \longleftrightarrow w_3$$

Proof



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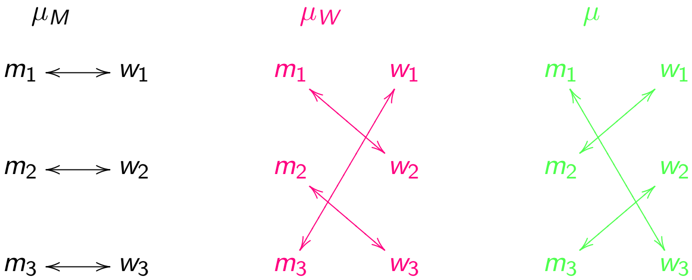

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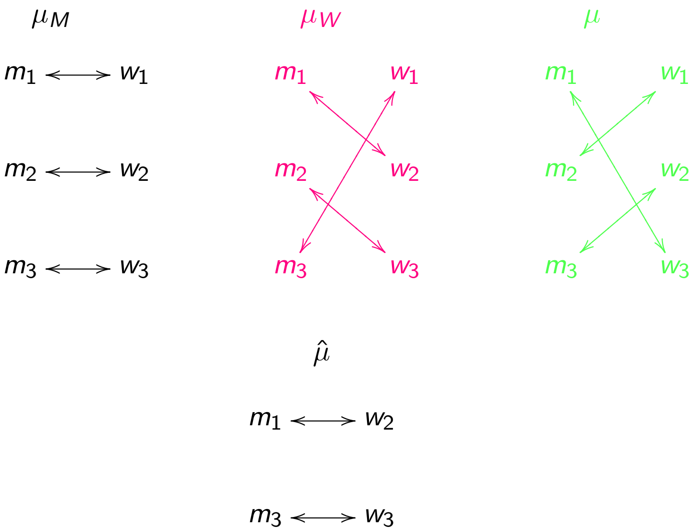
μ_M
m_1 w_1
m_2 w_2
m_3 w_3
μ_W
m_1 w_1
m_2 w_2
m_3 w_3
μ
m_1 w_1
m_2 w_2
m_3 w_3
$\hat{\mu}$
m_1 w_2
m_3 w_3

Proposition

If, for all m, $\mu_i(m) \neq \mu_j(m)$ *for all* $\mu_i, \mu_j \in H$*, then* $\mathcal{H}$ *is rationalizable.*

Proof.

m	w
$\mu_1(m)$	$\mu_n(w)$
$\mu_2(m)$	$\mu_{n-1}(w)$
$\vdots$	$\vdots$
$\mu_n(m)$	$\mu_1(w)$

$\square$

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So:

- Matching theory is refutable (everything is not rationalizable)
- Source of refutability is: $\mu(a) = \mu'(a)$ for some agents a.

Example

	m_1	m_2	m_3	m_4
μ_1	w_1	w_2	w_3	w_4
μ_2	w_1	w_3	w_4	w_2
μ_3	w_2	w_3	w_1	w_4

Can you find P s.t. μ_1, μ_2 and μ_3 are stable ?

Example

How do the m compare $\mu_1(m)$ and $\mu_2(m)$?

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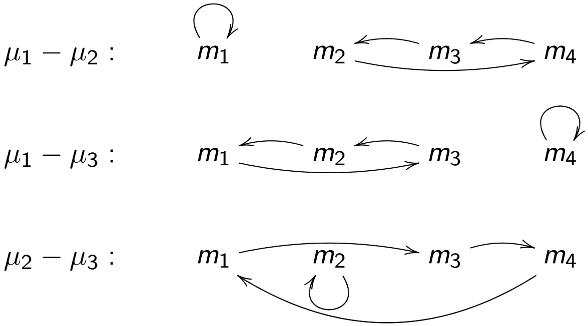
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Example (cont.)

all $m \in C = \{m_2, m_3, m_4\}$ agree on μ_1 and μ_2;
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Say $\mu_2(m)\, P(m)\, \mu_1(m)\ \forall m \in C$.

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But, $m_4 \in C$ and $\mu_1(m_4) = \mu_3(m_4)$ so $\mu_2(m)\, P(m)\, \mu_3(m)$ $\forall m \in C''$.

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Now: $m_1 \in C' \cap C''$, so

$$\mu_2(m_1)\, P(m_1)\, \mu_3(m_1)\, P(m_1)\, \mu_1(m_1).$$

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$\mathcal{H}$ is not rationalizable.

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Let $\mathbf{C}(\mu_i, \mu_j)$ the set of all connected components of $(M, E(\mu_i, \mu_j))$.

Female pairwise graphs

$(W, F(\mu_i, \mu_j))$

- vertex-set W
- $(w, w') \in F(\mu_i, \mu_j)$ if $\mu_j(w) = \mu_i(w)$.

Lemma

The following statements are equivalent:

1. *C is a connected component of $(M, E(\mu_i, \mu_j))$*
2. *$\mu_i(C)$ is a connected component of $(W, F(\mu_i, \mu_j))$*

In addition, if C is a connected component of $(M, E(\mu_i, \mu_j))$, then $\mu_j(C) = \mu_i(C)$.

Coincidence/conflict of interest

Lemma

Let $\mathcal{H}$ be rationalized by preference profile P. If $\mu_i, \mu_j \in \mathcal{H}$, and $C \in \mathbf{C}(\mu_i, \mu_j)$, then either (1) or (2) hold.

$$\mu_i(m)\, P(m)\, \mu_j(m) \forall m \in C \& \mu_j(w)\, P(w)\, \mu_i(w) \forall w \in \mu_i(C) \quad (1)$$

$$\mu_j(m)\, P(m)\, \mu_i(m) \forall m \in C \& \mu_i(w)\, P(w)\, \mu_j(w) \forall w \in \mu_i(C) \quad (2)$$

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Further, if P is a preference profile such that: for all $\mu_i, \mu_j \in \mathcal{H}$, and $C \in \mathbf{C}(\mu_i, \mu_j)$, either (1) or (2) hold, and in addition

$$\emptyset\, P(m)\, w \Leftrightarrow w \notin \{\mu(m) : \mu \in \mathcal{H}\}$$
$$\emptyset\, P(w)\, m \Leftrightarrow m \notin \{\mu(w) : \mu \in \mathcal{H}\},$$

then P rationalizes $\mathcal{H}$.

Lattice operations.

$C \in \mathbf{C}(\mu_i, \mu_j)$, either (3) or (4) must hold:

$$(\mu_i \wedge \mu_j)|_C = \mu_i|_C \text{ and } (\mu_i \vee \mu_j)|_C = \mu_j|_C \tag{3}$$

$$(\mu_i \wedge \mu_j)|_C = \mu_j|_C \text{ and } (\mu_i \vee \mu_j)|_C = \mu_i|_C. \tag{4}$$

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If H is rationalizable, cannot have

$$C \triangle C' \bigtriangledown C'' \triangle C''' \triangle C$$

Necessary Condition

Theorem

If $\mathcal{H}$ is rationalizable then $(\mathbf{C}, \mathbf{E}_{\triangle} \cup \mathbf{E}_{\bigtriangledown})$ can have no cycle with an odd number of $\bigtriangledown$.

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If $\mathcal{H}$ is rationalizable then $(\mathbf{C}, \mathbf{D}_{\triangle} \cup \mathbf{E}_{\bigtriangledown})$ can have no cycle with an odd number of $\bigtriangledown$.

Necessary and Sufficient Condition

Theorem

$\mathcal{H}$ is rationalizable if and only if $(\mathbf{C}, \mathbf{D}_{\triangle} \cup \mathbf{E}_{\bigtriangledown})$ has no cycle with an odd number of $\bigtriangledown$s, and for the resulting graph $(\mathbb{C}, \mathbb{D})$, there is a function $d : \mathbb{C} \to \{-1, 1\}$ that satisfies:

1. $\mathcal{C} \bigtriangledown \mathcal{C}' \Rightarrow d(\mathcal{C}) + d(\mathcal{C}') = 0$,
2. $(\mathcal{C}, \mathcal{C}', \mathcal{C}'') \in B \Rightarrow (d(\mathcal{C}) + d(\mathcal{C}'))\, d(\mathcal{C}'') \geq 0$.

Further, there is a rationalizing preference profile for each function d satisfying (1) and (2).

Identification

U_m is the set of women m is not matched to in any $\mu \in \mathcal{H}$.

Proposition

If $\mathcal{H}$ is rationalizable, then it is rationalizable by at least

$$(2|M|)^{|M|} \Pi_{m \in M} |U_m|$$

essentially different preference profiles.

Rationalizing Random Matchings

Proposition

If k is fixed,

$$\liminf_{n\to\infty} \mathbf{P}\left\{\mathcal{H}_k \text{ is rationalizable }\right\} \geq e^{-k(k-1)/2}$$

Precedent - I

Gale-Shapley-Conway: $S(P)$ is a NDL

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Blair: Any NDL is isomorphic to the core of some matching market

Precedent - II

Roth-Sotomayor:

> *We might hope to say something more about what kinds of lattices arise as sets of stable matchings, in order to use any additional properties thus specified to learn more about the market. (Blair's) Theorem shows that this line of investigation will not bear any further fruit.*