Fairness and efficiency for probabilistic allocations with endowments

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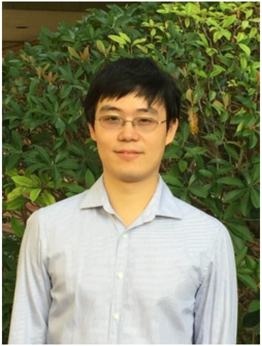
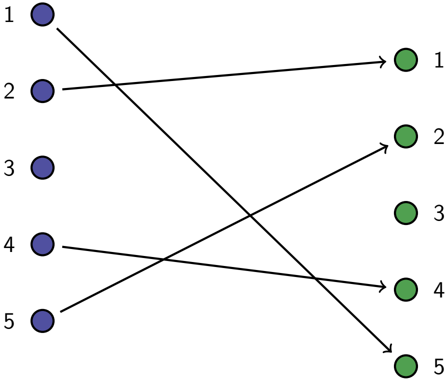
Antonio and Jun:

Discrete allocation



For example

- Jobs to workers
- Courses to students
- Organs to patients
- Schools to children
- Offices to professors.

Desiderata

- Efficiency
- Fairness
- Property rights.

Efficiency

Pareto optimality.

An assignment is efficient if there is no alternative (feasible) assignment that makes everyone better off and at least one agent strictly better off.

Fairness

Alice *envies* Bob at an assignment if she would like to have what Bob got.

An assignment is *fair* if no agent envies another agent.

Fairness

Fairness **requires** randomization.

If Alice and Bob want the same office $\Longrightarrow$ flip a coin.

Fairness vs. efficiency

When there is a conflict between efficiency and fairness, policy makers (and society?) often prioritize fairness.

Hence fairness is a priority in market design.

So we'll work with random assignments.

Pseudomarkets

Can we be fair and efficient?

Yes: use pseudomarkets

Pseudomarkets: Hylland and Zeckhauser 1979

Assign workers to jobs.

- L jobs.
- A *lottery*: $x^i = (x^i_1, x^i_2, \ldots, x^i_L)$
- $x^i_l =$ probability that i is assigned job l.

Pseudomarkets: Hylland and Zeckhauser 1979

Assign workers to jobs.

- L jobs.
- A *lottery*: $x^i = (x_1^i, x_2^i, \ldots, x_L^i)$
- $x_l^i =$ probability that i is assigned job l.
- utility function $u^i(x^i)$
- for ex. $u^i(x^i)$ can be an exp. utility.

Pseudomarkets

A lottery x^i satisfies

$$\sum_l x_l^i \leq 1$$

A lottery is an element of

$$\Delta_- = \{x \in \mathbf{R}_+^L : \sum_{j=1}^{L} x_j \leq 1\}$$

$u^i : \Delta_- \to \mathbf{R}$ (cont. & mon.)

Model

- Agents: $I = \{1, \ldots, N\}$.
- Objects: $S = \{s_1, \ldots, s_L\}$.
- $u^i : \Delta_- \to \mathbf{R}$ (cont. & mon.)

Allocations

An *allocation* is $x = (x^i)_{i=1}^N$, with $x^i \in \Delta_-^L$, s.t

$$\sum_{i \in I} x_s^i = 1$$

Fairness

i *envies* j at x if $u^i(x^j) > u^i(x^i)$

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$x^i = (1/L, \ldots, 1/L) \Longrightarrow$ no envy

Efficiency

An allocation x is *Pareto optimal* (PO) if there is no allocation y s.t

$$u^i(y^i) \geq u^i(x^i) \text{ for all } i \text{ and } u^j(y^j) > u^j(x^j)$$

for some j.

Hylland and Zeckhauser (1979)

Hylland and Zeckhauser (1979)

An *HZ-equilibrium* is a pair (x, p), with $x \in \Delta_{-}^{N}$ and $p = (p_s)_{s \in S} \geq 0$ s.t.

1. $\sum_{i=1}^{N} x^i = (1, \ldots, 1)$
2. x^i solves

$$\text{Max } \{u^i(z^i) : z^i \in \Delta_{-} \text{ and } p \cdot z^i \leq 1\}$$

Condition (1): supply = demand.
Condition (2): x^i is i's demand at prices p and income $= 1$.

Observe:

- Income is independent of prices
- Not a "closed" model (Monopoly money).

Fairness and efficiency

Suppose that each u^i is linear (expected utility).

Theorem (Hylland and Zeckhauser (1979))

There is a HZ equilibrium allocation. It is envy-free and PO.

This paper:

Fair assignment with endowments.

Why endowments?

- Endowments are relevant for *any* problem where we don't start from scratch.
- Existing allocation matters. Want agents to buy into market design, hence respect property rights.

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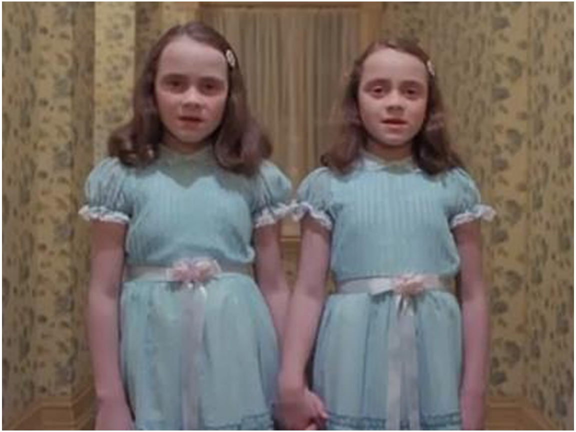
- Endowments are relevant for *any* problem where we don't start from scratch.
- Existing allocation matters. Want agents to buy into market design, hence respect property rights.
- School choice:
  - Property rights are captured by priorities.
  - As property rights, priorities are equivocal; not transparent.
  - Endowments are explicit property rights.
  - For ex., guarantee a:
    1. chance at a good school;
    2. neighborhood school;
    3. slot for a sibling.

This paper:

- Assignment with endowments
- Make agents *unequal*
- Confict between no-envy and property rights.

No envy: fairness for equals



Fairness for unequal agents?

- Agents have unequal endowments
- No envy may violate property rights.

This paper:

- We propose a notion of fairness for unequally endowed agents
- Prove it can be achieved with efficiency and individual rationality.
- Can be obtained as a market outcome.
- And respecting general constraint structures.

Related Literature

- Mkts. & fairness: Varian (1974), Hylland-Zeckhauser (1979), Budish (2011).
- Justified envy w/endowments: Yilmaz (2010)
- Allocations with constraints: Ehlers, Hafalir, Yenmez and Yildrim (2014), Kamada and Kojima (2015, 2017).

More references in the paper...

Fairness among unequals

- Each i has an *endowment* $\omega^i \in \Delta$.
- ω^i is an initial lottery.
- Suppose that $\sum_i \omega^i = (1, \ldots, 1)$.

For example, suppose schools are allocated via a lottery. Admission probabilities reflect: neighborhood school (walk-zone priority), sibling priority, or test scores.

Model

- Agents: $I = \{1, \ldots, N\}$.
- Objects: $S = \{s_1, \ldots, s_L\}$. Suppose $N = L$.
- For each $i \in I$,
  - $u^i : \Delta_- \to \mathbf{R}$
  - $\omega^i \in \Delta$.
- $\sum_i \omega^i = (1, \ldots, 1)$.

Walrasian equilibrium

A *Walrasian equilibrium* is a pair (x, p) with $x \in \Delta_-^N$, $p \geq 0$ s.t

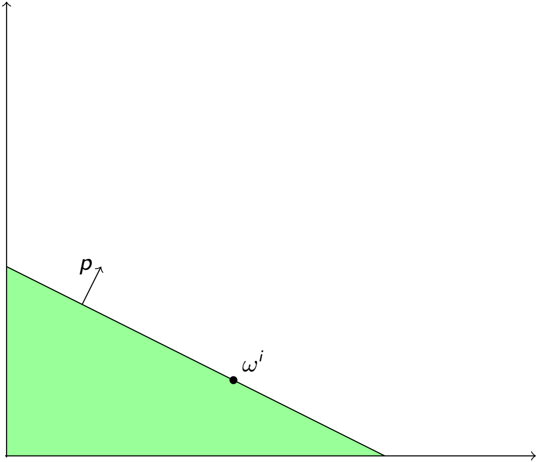
1. $\sum_{i=1}^N x^i = \sum_{i=1}^N \omega^i$; and
2. x^i solves

$$\text{Max } \{u^i(z^i) : z^i \in \Delta_- \text{ and } p \cdot z^i \leq p \cdot \omega^i\}$$

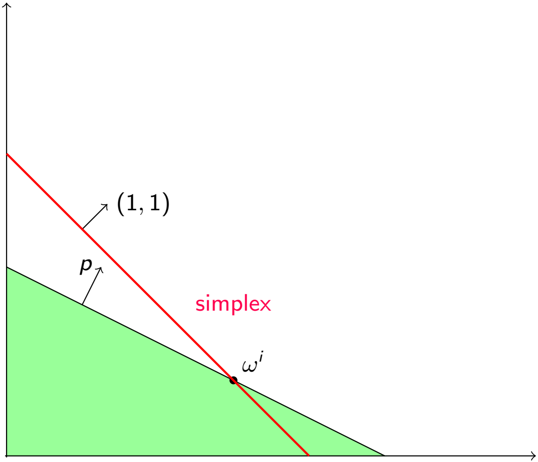
Proposition (Hylland and Zeckhauser (1979))

There are economies in which all agents' utility functions are expected utility, that posses no Walrasian equilibria.

Budget set



Budget set



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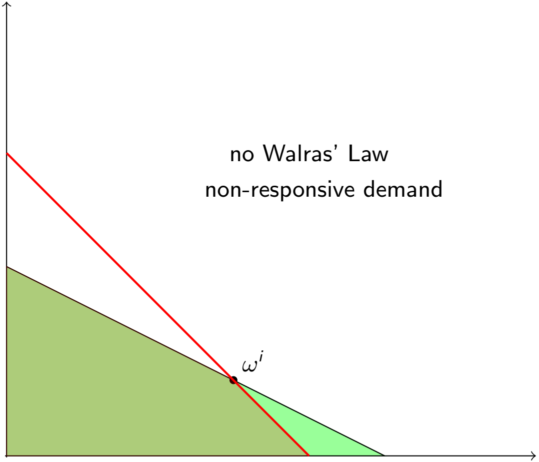

HZ Example

3 agents; exp. utility

	u^1	u^2	u^3
s_A	10	10	1
s_B	1	1	10

Endowments: $\omega^i = (1/3, 2/3)$.

HZ Example

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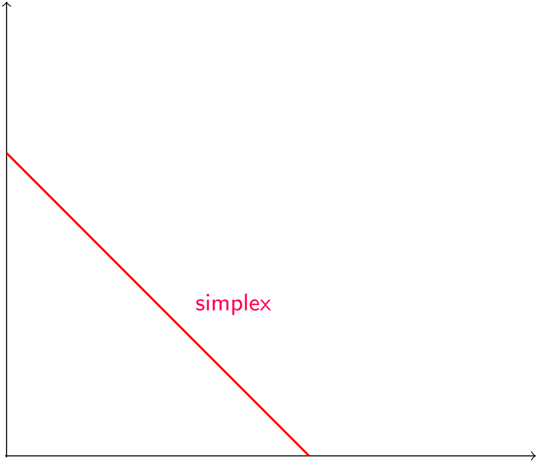
Endowments: $\omega^i = (1/3, 2/3)$.

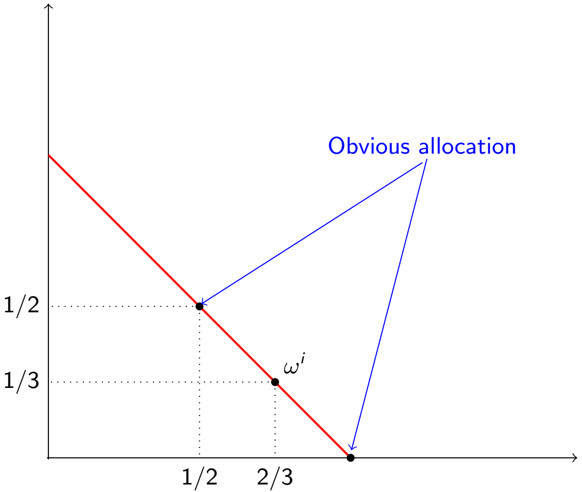
Obvious allocation:

$$x^1 = x^2 = (1/2, 1/2)$$
$$x^3 = (0, 1)$$

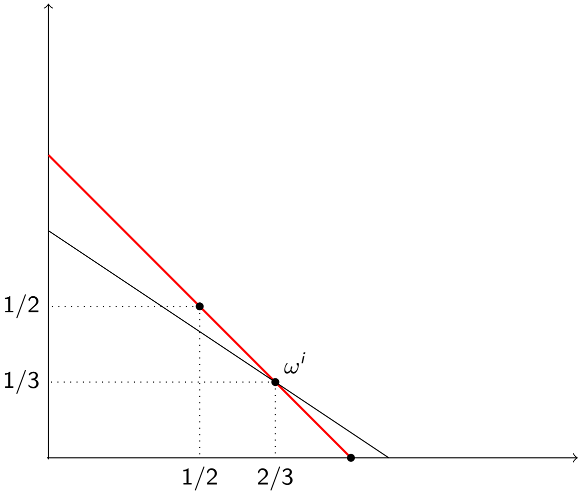
HZ Example



HZ Example



HZ Example



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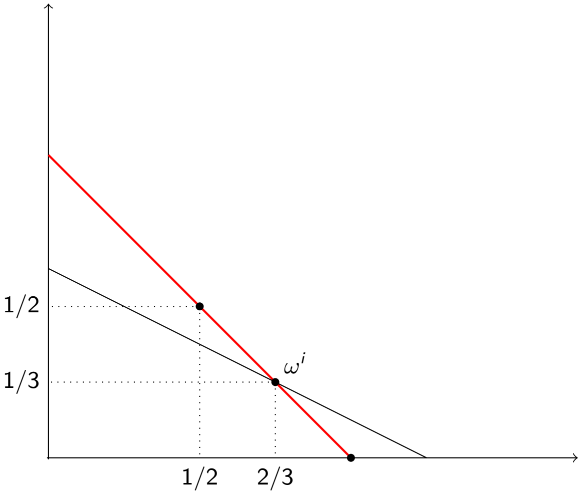
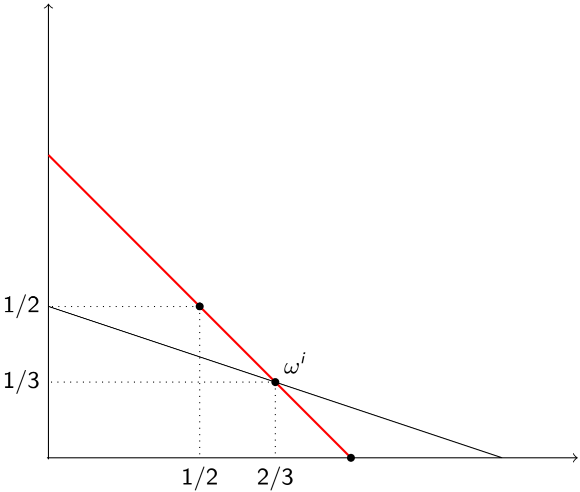


HZ Example



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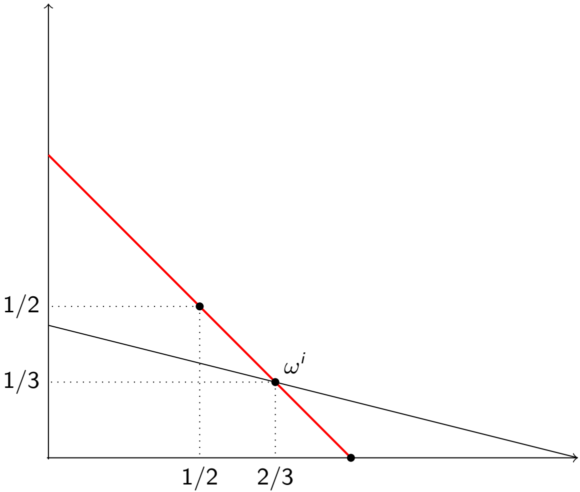


Moreover, . . .

- the first welfare theorem fails.
- There are Pareto ranked Walrasian equilibria.

Our results

Let x be an allocation.

- x is *weak Pareto optimal* (wPO) if $\nexists$ an allocation y s.t $u^i(y^i) > u^i(x^i)$ for all i
- *ε-weak Pareto optimal* (ε-PO), for $\varepsilon > 0$, if $\nexists$ an allocation y s.t $u^i(y^i) > u^i(x^i) + \varepsilon$ for all i.

Let x be an allocation.

- x is *acceptable* to i if $u^i(x^i) \geq u^i(\omega^i)$.
- x is *individually rational* (IR) if it is acceptable to all agents.

Justified envy

i envies j at x if $u^i(x^j) > u^i(x^i)$.

Such envy will be tolerated (i.e not be justified) only if j's endowment is "good enough."

Justified envy

i envies j at x if $u^i(x^j) > u^i(x^i)$.

Such envy will be tolerated (i.e not be justified) only if j regards x^i as *unacceptable.*

Justified envy

i envies j at x if $u^i(x^j) > u^i(x^i)$.

Such envy will be tolerated (i.e not be justified) only if $u^j(\omega^j) > u^j(x^i)$

Justified envy: defn

i has *justified envy* towards j at allocation x if

$$u^i(x^j) > u^i(x^i) \text{ and } u^j(x^i) \geq u^j(\omega^j).$$

Fairness

Let x be an allocation.

x has *no justified envy* (NJE) if no agent has justified envy towards any other agent at x.

Fairness

Observe: NJE and IR imply *equal treatment of equals*.

Fairness

Let x be an allocation.

x has *no justified envy* (NJE) if no agent has justified envy towards any other agent at x.

Justified envy

- i has *strong justified envy* (SJE) towards j at x if $u^i(x^j) > u^i(x^i)$ and $u^j(x^i) > u^j(\omega^j)$.
- For $\varepsilon > 0$, i has *ε-justified envy* (ε-JE) towards j at x if $u^i(x^j) > u^i(x^i)$ and $u^j(x^i) > u^j(\omega^j) - \varepsilon$.

Justified envy

no ε-justified envy $\implies$ no justified envy $\implies$ no strong just. envy

Fairness, property rights and efficiency

Theorem

Suppose utility functions are concave.

1. $\exists$ *an allocation that is* ε*-IR,* ε*-PO and has no* ε*-justified envy;*
2. $\exists$ *an allocation that is IR, wPO and has no strong justified envy.*
3. *Moreover, if utility functions are expected utility* $\exists$ *an allocation that is IR, PO and has no strong justified envy.*

Theorem

Suppose utility functions are quasi-concave, and that

♠

Then there exists continuous functions $m^i : \Delta \to \mathbf{R}_+$ *and* $(x, p) = ((x^i)_{i=1}^I, p) \in (\Delta_-^I) \times \Delta$*, such that*

1. $\sum_i x^i = \sum_i \omega^i$ *(x is an allocation; or, "supply equals demand").*
2. *x is Pareto optimal, individually rational and has no justified envy.*
3. $x^i \in \textit{argmax}\{u^i(z^i) : z^i \in \Delta_- \text{ and } p \cdot z^i \leq m^i(p)\}$

Constraints

- Given as primitive a set $\mathcal{A}^{\mathcal{C}}$ of allocations.
- The *feasible* allocations.
- Assume $\mathcal{A}^{\mathcal{C}}$ is convex and compact.

For example:

- Distributional constraints.
- Geographical constraints.
- etc.

Constraints

i has an *justified envy* towards j at an allocation $x \in \mathcal{A}^{\mathcal{C}}$ if

$$u^i(x^j) > u^i(x^i),\ u^j(x^i) \geq u^j(\omega^j) \text{ and } x_{i \leftrightarrow j} \in \mathcal{A}^{\mathcal{C}}.$$

Constraints

$i, j \in I$ are of *equal type* if

$$\text{for all } x \in \mathcal{A}^{\mathcal{C}},\ x_{i \leftrightarrow j} \in \mathcal{A}^{\mathcal{C}}.$$

Constraints

Theorem

Suppose agents' utility functions are concave and that $\omega \in \mathcal{A}^{\mathcal{C}}$.

1. *For any $\varepsilon > 0$, there exists an allocation that is ε-IR, ε-PO and has no equal-type ε-justified envy;*
2. *There exists an allocation that is IR, wPO, and has no strong equal-type justified envy.*

Ideas

Theorem

Suppose utility functions are concave.

1. $\exists$ *an allocation that is* ε*-IR,* ε*-PO and has no* ε*-justified envy;*
2. $\exists$ *an allocation that is IR, wPO and has no strong justified envy.*
3. *Moreover, if utility functions are expected utility* $\exists$ *an allocation that is IR, PO and has no strong justified envy.*

Idea

Consider problem

$$\text{Max } \sum_i \lambda_i u^i(x_i)$$

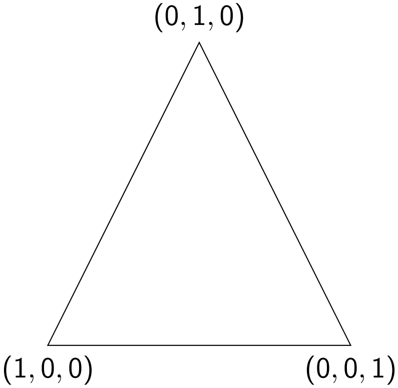
s.t. x is an allocation.

Obtain a NJE allocation from this problem

by choosing right welfare weights, $(\lambda_i) \in \Delta^N$.

(Actual proof uses an approximation to this problem, hence the ε).

KKM Lemma



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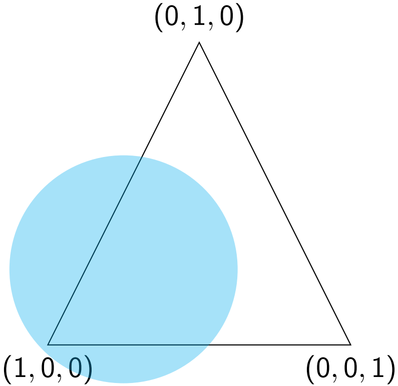

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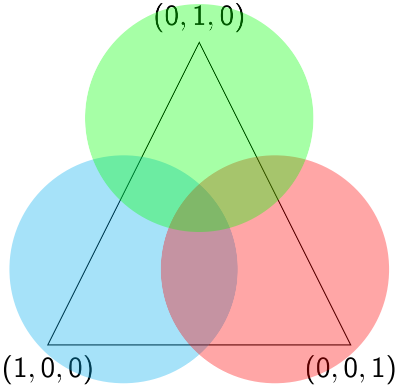

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1. $\sum_i x^i = \sum_i \omega^i$ *(x is an allocation; or, "supply equals demand").*
2. x *is Pareto optimal, individually rational and has no justified envy.*
3. $x^i \in argmax\{u^i(z^i) : z^i \in \Delta_-$ *and* $p \cdot z^i \leq m^i(p)\}$

♠: $\exists l$ s.t. for any $i \in I$ and $x^i \in \Delta_-^L$, decreasing consumption of any object $k \neq l$ in favor of l leads to an increase in u^i; and $\omega_l^i > 0$.

$$e^i(v,p) = \inf\{p \cdot x : u^i(x) \geq v\},$$

for $p \in \Delta^L$ and $v \in \mathbf{R}$.

Let $v^i = \sup u^i(\Delta^L_-)$ be the utility of agent i when she is satiated.

For any scalar $m \geq 0$ and $p \in \Delta^L$, let

$$\mu^i(m, p) = \text{median}(\{e^i(u^i(\omega^i), p), m, e^i(v^i, p)\}).$$

Consider the function

$$\varphi(m, p) = \sum_i \mu^i(m, p) - \sum_i p \cdot \omega^i.$$

Observe that

- $e^i(u^i(\omega^i), p) \leq e^i(v^i, p)$.
- μ^i is continuous and $m \mapsto \mu^i(m, p)$ weakly monotone increasing.
- φ is continuous and $m \mapsto \varphi(m, p)$ weakly monotone increasing.
- $\varphi(m, p) \leq 0$ for $m \geq 0$ small enough as $e^i(u^i(\omega^i), p) \leq p \cdot \omega^i$.

- in the case that $\sum_i e^i(v^i, p) < \sum_i p \cdot \omega^i$, we let $m^i(p) = e^i(v^i, p)$.
- in the case that $\sum_i e^i(v^i, p) \geq \sum_i p \cdot \omega^i$, we have that $\varphi(m, p) \leq 0$ for $m \geq 0$ small enough, and $\varphi(m, p) \geq 0$ for $m \geq 0$ large enough. So $\exists\ m^* \geq 0$ with $\varphi(m^*, p) = 0$. Now let $m^i(p) = \mu^i(m^*, p)$.

No justified envy

Suppose that i envies j at x^*. This implies that i is not satiated, hence $m^i(p^*) < e^i(v^i, p^*)$.
It also implies that $m^i(p^*) < m^j(p^*)$ as $m^i(p^*) < p^* \cdot x^j = m^j(p^*)$.
On can then show that, by defn. of m^j, $m^j(p^*) = e^j(u^j(\omega^j), p^*)$.

No justified envy

We obtain that

$$p^* \cdot x^i = m^i(p^*) < m^j(p^*) = e^j(u^j(\omega^j), p^*),$$

and hence $u^j(x^i) < u^j(\omega^j)$ by definition of expenditure function. So i's envy is not justified.