Stable allocations in discrete economies

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Abstract

We study discrete allocation problems, as in the textbook notion of an exchange economy, but with indivisible goods. The problem is well-known to be challenging. The model is rich enough to encode some of the most pathological bargaining models in game theory, like the roommate problem. Our contribution is to show the existence of stable allocations (outcomes in the weak core, or in the bargaining set) under different sets of assumptions. Specifically, we consider dichotomous preferences, categorical economies, a gains from trade property, and discrete TU markets. The techniques used are varied, from Scarf's balancedness condition, to a generalization of the TTC algorithm by means of Tarski fixed points.

1 Introduction

Economists have a good understanding of discrete allocation problems under some popular, but very special, assumptions. The literature with quasi-linear preferences, or transfers, is highly developed. Almost everything we know about discrete allocation problems is in the literature on auctions and pricing, under the assumption of quasi-linear preferences. The literature

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refers to these models as markets "with money," and money is ubiquitous in mechanism design. Without transfers, most progress is limited to models of unit demand; the so-called housing market of Shapley and Scarf (1974).[1] The general discrete multi-good allocation problem is, however, known to be very difficult. The present paper is an attempt at furthering our understanding of this difficult problem.

The discrete allocation problem is important for conceptual and theoretical reasons, and because it covers important practical applications. First, theory. The discrete allocation problem is the text-book model of an exchange economy, but without the technical assumption of infinitely divisible goods. Indeed, in an economy with infinitely divisible goods, standard assumptions of convexity and continuity suffice to establish the existence of various solution concepts (see chapters 15-17 in Mas-Colell et al. (1995)). Many important questions about the structure of equilibria, connection between different solution concepts and their welfare implications, as well as the scope of general equilibrium theory, are all well understood. Without the assumption of infinitely divisible goods, much less is known about the basic model of exchange. So we think that it is conceptually very important to better understand models of discrete multi-goods markets. Put simply (if dramatically): *The profession's understanding of markets and exchange is limited by the extent of our understanding of the general model with indivisible goods.*

Pure theory aside, some important applications rely on a better understanding of the general discrete assignment problem. Perhaps the most glaring class of applications are multi-item auctions for agents that are not risk-neutral over monetary transfers. The most successful practical application is probably to course bidding: Krishna and Ünver (2008); Sönmez and Ünver (2010a), Budish and Cantillon (2012), Budish, Cachon, Kessler, and Othman (2017), and Othman et al. (2010).

1.1 Overview of results

The thrust of our paper is to propose a series of sufficient conditions that ensure that some notion of stability exists in discrete multi-good exchange economies. The question of the existence and structure of competitive equilibria is obviously important, but not the focus of our paper. We deal exclu-

[1] We focus on one-sided, object allocation, models. Not on two-sided markets.

sively with game-theoretic bargaining paradigms (within discrete exchange economies) and discuss variations of core stability.

The techniques that we use are varied: two of our results reduce the problem to Scarf's balancedness condition (Scarf (1967)). One result connects the model to the literature on NTU convex games (a generalization of the TU games with supermodular characteristic functions of Shapley (1971)). Our final result depends on a Tarski fixed-point argument, and a generalization of the TTC algorithm from the housing model to a discrete exchange economy.

We proceed with an informal description of our main results:

1) Economies with dichotomous preferences have nonempty weak core (Theorem 1). We assume that each agent classifies items into good/bad, or acceptable/unacceptable. Agents want as many acceptable objects as possible, meaning that the value of a bundle is equal to the sum of acceptable items that it contains. Under this structure, we show that the game satisfies Scarf's balancedness condition, and therefore the core is nonempty.[2]

2) Categorical economies. Goods are grouped into categories, and agents may consume at most one good of each category: The *house-car-boat model*, to use the language of Moulin (2014). Agents have additively separable and dichotomous utilities over categorical consumption. Under these assumptions, Theorem 2 states that the weak core is nonempty. The proof is again through Scarf's balancedness condition, and it features a rounding algorithm — in that sense our proof is algorithmic, even though the existence of a core allocation does not rest on a constructive argument.

3) Gains from trade. We consider economies with injective utilities, thus moving away from the dichotomous assumption in our first results. We introduce a "gains from trade" assumption that says that, when two coalitions may, through trading, achieve certain utilities, then the union of all agents in the two coalitions can do no worse by trading jointly. The assumption may sound quite general, but it rules out the exclusive uses of a resource, as we explain in Section 3.3. Theorem 3 shows that an economy with injective utilities and gains from trade has a nonempty core. The result follows again from Scarf's balancedness condition, but there an alternative proof that proceeds by showing that the economy may be associated to a convex NTU game, which are known to possess a nonempty weak core.

[2]Rather famously, this is also how Shapley and Scarf (1974) proved that the core is nonempty in the housing model. David Gale pointed out that their result could be greatly simplified by means of the Top Trading Cycle (TTC) algorithm . The TTC, however, does not have a straightforward extension beyond the housing model.

4) A generalization of the Top Trading Cycles (TTC) algorithm. Our final set of results rest on an algorithmic formulation of the problem. The housing model is solved using the TTC algorithm: Borrowing ideas from the two-sided matching literature, we propose an algorithm that generalizes the TTC (while seemingly being very different, we show that our algorithm replicates the steps of the TTC when applied to the housing model). Under the assumptions of injectivity, monotonicity, and "discrete transferable utility," we show (Theorem 4) that the algorithm finds an allocation in the pairwise bargaining set.

Our last result merits a few remarks. Discrete transferable utility means that any two agents' Pareto frontier from trading has a "discrete slope" of -1. This assumption is interesting because it allows for an existence result that does not explicitly use transfers, or the convexity and continuity properties associated with the use of a continuum numeraire, but where the Pareto frontier for pairs of agents is reminiscent of models with transfers. The techniques involved are completely different from what the standard models with transfers normally require.

The other comment is that, in our generalization of the TTC, we focus on a farsighted weakening of the core: blocks may be disregarded when agents know that some of those who are involved in blocking will later have incentives to renege from the blocking action. If we think of blocks as objections, then the bargaining set (Aumann and Maschler (1964)) is the set of allocations for which any objections are subject to counterobjections. As far as we know, ours is the first paper to study the bargaining set in discrete multi-goods exchange economies. In contrast, the bargaining set has been studied extensively in the model with infinitely divisible goods (see, for example, Mas-Colell (1989a); Zhou (1994); Anderson et al. (1997)).

1.2 Related Literature

Shapley and Scarf (1974) introduce the model of a housing market, which has been studied very extensively. It is a special case of our model, when agents have unit demands and are endowed with a single good. Their existence proof relies on Scarf's sufficient condition, but they note that a simpler and constructive argument is possible by means of Gale's Top Trading Cycles (TTC) algorithm. The literature following up on Shapley-Scarf, and analyzing the housing market, is huge. Roth and Postlewaite (1977), for example, discuss the differences between strong and weak blocks, and clarify the relation between the core of the housing market and the competitive equilibrium allocations. Ma (1994) and Sönmez (1999) study the incentive properties of core allocations. Sönmez and Ünver (2010b) and Roth, Sönmez, and Ünver (2004) apply the model in practical market-design settings.

The literature on discrete exchange economies is also significant. With no pretence of going through an exhaustive review, we can mention that Henry (1970) is mainly focused on the existence problem for competitive equilibrium. He shows that, unless very restrictive assumptions are imposed, an equilibrium is not going to exist. A number of papers seek to overcome these negative results by considering discrete economies in which there exists one perfectly divisible goods, a *numeraire*: Perhaps the first paper in this setting is Mas-Colell (1977), who proposes a model in which he can show the existence of competitive equilibria. Quinzii (1984) shows that the core is nonempty in a housing model with unit demand, as well as proving a core equivalence theorem. Svensson (1983), also imposing unit demand, obtain existence of allocation with various equilibrium and normative properties. More recently, Baldwin, Edhan, Jagadeesan, Klemperer, and Teytelboym (2020) exhibit a connection between a model with transferable utility, and a general model of a market with income effects, which allows them to obtain existence results. And Jagadeesan and Teytelboym (2021) proves the existence of quasi-equilibrium allocations, which are then exploited to show that the set of stable allocations are nonempty.

Another line of attach has focused on large economies: keeping the number of goods fixed, while letting the number of agents grow large. See Starr (1969) and Dierker (1971). Mas-Colell (1977) works in the continuum limit (and also, in fact, assumes a perfectly divisible good). More recently, Budish (2011) considers discrete markets and shows that existence and incentive properties are alleviated when the number of agents is large. See also Budish, Cachon, Kessler, and Othman (2017) for a real-world implementation of these ideas to course bidding in business schools.

The model of a categorical economy that we cover in Section 3.2 is introduced by Moulin (2014), who presents as an open question the determination of whether the core is empty. Konishi, Quint, and Wako (2001) provide an answer in the form of a non-existence example (we revisit this example in Section 3.2). Our contribution is to find sufficient conditions on a categorical economy, namely dichotomous preferences for each category of good, under which the core is nonempty. Closer in spirit to our exercise, Inoue (2008) provides a discrete convexity assumption (essentially that agents' upper contour sets are convex, up to the discreteness inherent in the problem) and shows that the weak core is nonempty.

The literature on the bargaining set started with Aumann and Maschler (1964), and includes several different variations on the notion of a bargaining set, such as Mas-Colell (1989b) and Zhou (1994). Ours is a variation of the bargaining set adapted to the application to discrete economies. There are known, general, game-theoretic existence results, such as those of Peleg (1967) in the setting of transferable utility, and Peleg (1963) or Vohra (1991) for games without transfers. Our model does not satisfy the assumption in these papers: we do not have transferable utility, nor are the convexity assumptions in Peleg (1963), or the balancedness condition of Vohra (1991), satisfied. We also want to emphasize the algorithmic and constructive nature of our existence result for the bargaining set, while the existing literature often uses non-constructive, topological, fixed-point arguments.

2 Model

An *economy* is a tuple $E = (O, \{(v_i, \omega_i) : i \in A\})$ in which

- O is a finite set of *objects*;
- A is a finite set of *agents*;
- each agent $i \in A$ is described by a utility function $v_i : 2^O \to \mathbf{R} \cup \{-\infty\}$ and a nonempty endowment $\omega_i \subseteq O$, with $O = \cup_i \omega_i$ and $\omega_i \cap \omega_j = \emptyset$ when $i \neq j$.

We allow utilities to take on the value $-\infty$ in order to encode agents' consumption space through the domain of the utility function. For example, we may consider a *shoe economy* in which agents desire a pair of left and right shoes. Then we may have three pairs of shoes: $O = \{\ell_1, \ell_2, \ell_3, r_1, r_2, r_3\}$, where ℓ_k is a left shoe and r_k a right one. Now we can say, for example, that $v_i(\{\ell_1, r_2\}) = 10 > 7 = v_i(\{\ell_3, r_1\})$, while $v_i(\{\ell_1, \ell_2\}) = -\infty$ says that consuming two left shoes is not allowed in the model.

More conventionally, the housing model of Shapley and Scarf (1974) is a special case of our model: Suppose that $O = \{h_i : i \in A\}$ contains exactly one house, h_i, for each agent i in A; suppose that $\omega_i = \{h_i\}$, so that i owns the ith house, and let $v_i(X) = -\infty$ when X is not a singleton subset of O.

We sometimes impose additional conditions to the utility functions. A utility v_i is *strictly monotone* if $X \subsetneq X'$ implies that $v_i(X) < v_i(X')$; and it is *injective* if $X \neq X'$ implies that $v_i(X) \neq v_i(X')$.[3]

Remark. We assume strictly monotone preference for some of our results, but it should be clear that the assumption as stated is stronger than needed. For example, for our purposes, we may think of the housing model as having strictly monotone utilities, even though it fails the definition given above. The key assumption would be strict monotonicity on the appropriate domain of agents' utilities. The housing model's unit demand assumption violates the definition of strict monotonicity per se, but such violation is immaterial for our purpose as each agent obtains exactly one house in any individually-rational exchange.

The following definitions are standard: An *allocation* in the economy E is a pairwise-disjoint collection of sets of objects, $\{X_i : i \in A\}$, with the property that $\cup_i X_i \subseteq O$. A nonempty subset $S \subseteq A$ is termed a *coalition*. An S-allocation, for a coalition S, is a pairwise-disjoint collection of sets of objects $\{X_i : i \in S\}$ with the property that $\cup_{i\in S} X_i \subseteq \cup_{i\in S}\omega_i$. We think of an allocation as the outcome if exchange among the agents in the economy, and of an S-allocation as the outcome of exchange among the members of the coalition S.

Given an allocation $X = \{X_i\}$ we say that the coalition S *weakly blocks* X if there exists an S-allocation $\{X'_i : i \in S\}$ with $v_i(X'_i) \geq v_i(X_i)$ for all $i \in S$, and $v_i(X'_i) > v_i(X_i)$ for at least one $i \in S$. In contrast, S *strongly blocks* X if there exists an S-allocation $\{X'_i : i \in S\}$ with $v_i(X'_i) > v_i(X_i)$ for all $i \in S$.

The *weak core* of the economy E is the set of allocations that are not strongly blocked by any coalition, while the *strong core* is the set of allocations that are not weakly blocked.

Our first example illustrates the difference between the weak and the strong core, and shows how the strong core may be a proper subset of the weak core, even when utilities are injective.

Example 1 (Weak core $\neq$ strong core with injective utilities.)**.** *Consider a shoe economy with three agents:* $A = \{1, 2, 3\}$ *and endowments* $(\{\ell_i, r_i\})_{i=1,2,3}$.

[3]Allowing for indifference presents problems, even in the simple housing market. Wako (1991) shows that the core may be empty in the Shapley-Shubik housing model as soon as agents may be indifferent between different houses. Some of our results will allow for indifferences, but others will impose "injective" utilities.

Assume ordinal preferences as follows:

Agent 1	*Agent 2*	*Agent 3*
$(\ell_2, r_3)^X$	$(\ell_3, r_1)^X$	$(\ell_1, r_2)^{X,Y}$
$(\ell_3, r_1)^Y$	$(\ell_2, r_3)^Y$	(ℓ_3, r_3)
(ℓ_1, r_1)	(ℓ_2, r_2)	(ℓ_3, r_3)
$\dots$	$\dots$	$\dots$

The table uses superscripts to identify two allocations. The first allocation is $X_1 = (\ell_2, r_3)$, $X_2 = (\ell_3, r_1)$, and $X_3 = (\ell_1, r_2)$. The second is $Y_1 = (\ell_3, r_1)$, $Y_2 = (\ell_2, r_3)$ and $Y_3 = (\ell_1, r_2)$. Now it is easy to see that the strong core consists only of allocation X, while the weak core contains both. Key here is that, in using X to block Y, agents 1 and 2 need to trade items that belong to agent 3, which requires 3's "permission." A weak block only requires 3's weak preference for them to grant permission, while the strong block insists on a injective preference.

In the example, agents have preferences that display no indifferences, which may be represented by a injective utility.

The weak core may be empty. Our second example exhibits a shoe economy with an empty core.

Example 2 (Empty weak core in a shoe economy.). *$A = \{1, 2, 3\}$ with endowment $\omega_i = (\ell_i, r_i)_{i=1,2,3}$. Assume preferences such that*

Agent 1	*Agent 2*	*Agent 3*
$(\ell_1, r_2)^X$	$(\ell_2, r_3)^Y$	$(\ell_1, r_3)^Z$
$(\ell_3, r_1)^Z$	$(\ell_2, r_1)^X$	$(\ell_3, r_2)^Y$
$(\ell_1, r_1)^Y$	$(\ell_2, r_2)^Z$	$(\ell_3, r_3)^X$
$\dots$	$\dots$	$\dots$

The table depicts the bundles that each agent regards as at least as good as their endowment, and exhibits three allocation by means of superscripts. For example the allocation $(X_1, X_2, X_3) = ((\ell_1, r_2), (\ell_2, r_1), (\ell_3, r_3))$ results from agents 1 and 2 trading right shoes, while agent 3 consumes her endowment.[4]

Now it is easy to verify that the weak core is empty. The first allocation is blocked by agents 2 and 3, using the second allocation (which is really an

[4]Trading of right and left shoes might suggest a version of the TTC for each type of shoe. This approach turns out not to work. It does not produce a core allocation, even for economies with additively separable preferences over shoes.

$\{2,3\}$*-allocation). The second allocation is blocked by agents 1 and 3 using the third allocation, which is in turn blocked by agents 1 and 2 by means of allocation 1.*

Some of our results will involve limits on the size of the possible blocking coalitions. An allocation is *pairwise stable* if it is individually rational and not weakly blocked by any coalition of size two. The motivation for considering such laxer stability concepts is that it may be difficult for agents to coordinate blocks among large coalitions. Pairwise stability is the most basic collective bargaining model, focusing on the smallest possible non-trivial blocking coalitions.

When we want to talk loosely, and informally, about allocations that are robust to blocking by some family of coalitions, we shall simply call them *stable*.

Our last example is meant to illustrate the seriousness of the non-existence problem. We argue that the same difficulties in establishing core allocations in the roommate problem, a problem for which stable allocations are known not to exist, are present in the discrete exchange economies that are the object of our paper. The problem we are facing is therefore, in a sense, harder than the roommate problem.

Example 3. *Consider an instance of the roommate problem. There are three agents, who are meant to pair up in couples or be left to live alone. Their preferences over roommates are given by the next table.*

Agent 1	*Agent 2*	*Agent 3*
2	3	1
3	1	2
1	2	3

It should be clear from the table that no configuration of couples and singles is stable. For example, if agent 1 pairs up with 2, and we leave agent 3 living alone, then 2 and 3 can block by pairing up instead. The core of the NTU game defined by the roommate problem in this example is empty. In fact, this example is not at all hard to come by. Stability in the roommate problem is famously challenging.

Why do we bring up the roommate problem? Because the logic behind non-existence of stable outcomes in the roommate problem above is the same *as in our prior "shoe economy" example, Example 2. In the latter example,* *value could only be generated by trading in pairs. Agent 1 wants to trade with 2, but if they trade then 3 is left consuming their endowment. Then agent 2, in turn, would prefer to trade with 3, who would agree. Example 2 illustrates that the model of a discrete exchange economy is rich enough to encode some of the best-known pathological situations in game theory.*

3 Results

In light of our previous discussion and examples, it is clearly impossible to obtain a general result about the existence of the core. The model is arguably rich enough to replicate any pathological behavior that can be exhibited in a NTU game; such as the non-existence of stable outcomes in the roommate problem, as discussed in our last example. The situation is, however, not hopeless. The model is also rich enough that there is space to add structure in ways that can ensure existence. That added structure is, indeed, the name of the game in the rest of the paper. We propose various sufficient conditions on the primitives of an economy that allow us to prove the existence of stable outcomes. The focus in this section is on the weak core, and on sufficient conditions that allow us to apply Scarf's theorem to show existence. In the next section we turn to other techniques, and to the bargaining set as a solution.

3.1 Dichotomous preferences.

An economy $E = (O, \{(v_i, \omega_i) : i \in A\})$ has *dichotomous item preferences* if, for each agent i there is a set of objects G_i so that $v_i(X_i) = |X_i \cap G_i|$.

Models of markets and social choice with dichotomous preferences have been studied quite extensively; see, for example, Bogomolnaia and Moulin (2004), Bogomolnaia, Moulin, and Stong (2005), and Aziz, Bogomolnaia, and Moulin (2019). In our version of the model, each agent classifies items into "acceptable" and "unacceptable" goods, and evaluate a bundle by the number of acceptable goods that it contains. Our first results shows that the weak core is non-empty. The proof, which can be found in Section 5, proceeds by showing that dichotomous preferences imply the sufficient condition of Scarf (1967). (This, in turn, involves a basic rounding procedure that resembles the usual proof of the Birkhoff-von Neumann theorem.)

Theorem 1. *An economy with dichotomous item preferences has nonempty weak core.*

The proof of Theorem 1 can be found in Section 5. In fact, all proofs are collected in Sections 5 and 6.

3.2 Categorical economies

An economy $E = (O, \{(v_i, \omega_i) : i \in A\})$ is *categorical* if there exists a number K of *categories of objects*, and sets of objects O^k, for each category $k \in \{1 \dots K\}$; so that 1) agents may consume at most one object of each category, and 2) agents' utility is dichotomous in each category, and 3) additively separable over categories. Formally, each agent i is endowed with a utility $v_i^k : O^k \to \{0, 1\}$ for each category k, such that $v_i(\emptyset) = 0$; and $v(X) \equiv \sum_k v_i^k(X \cap O^k)$. Here we adopt the convention that $v_i^k(X \cap O^k) = 1$ if there is some object $o \in X \cap O^k$ with $v_i^k(o) = 1$, and 0 otherwise.

As mentioned in the introduction, Moulin (2014) propose the model of a categorical economy (a "house-car-boat economy") and leaves open the question of whether the core is empty. We provide the first positive answer, under the additive separability and dichotomous assumptions.

Theorem 2. *A categorical economy has nonempty weak core.*

One might want to extend Theorem 2 to non-dichotomous utilities over each category, but the following example shows that this is not possible.

Example 4. *Consider an economy with $K = 2$ categories, in fact a left-right shoe economy. Suppose that there are four agents with ordinal preferences given as follows:*

$$\begin{array}{l}
(l_4, r_1) \succ_1 (l_4, r_2) \succ_1 (l_1, r_1) \succ_1 \dots \\
(l_2, r_1) \succ_2 (l_1, r_1) \succ_2 (l_2, r_3) \succ_2 (l_2, r_2) \succ_2 \dots \\
(l_3, r_4) \succ_3 (l_3, r_2) \succ_3 (l_4, r_4) \succ_3 (l_3, r_3) \succ_3 \dots \\
(l_3, r_4) \succ_4 (l_3, r_3) \succ_4 (l_2, r_4) \succ_4 (l_4, r_4) \succ_4 \dots
\end{array}$$

This example coincides with Example 2.3 in Konishi et al. (2001), who show that the weak core in this game is empty. We do not go over the argument for why the core is empty, and refer the reader to Konishi et al. for details.

It turns out that there is a cardinal utility representation for these preferences so that each agent has additively-separable utility over pair of objects of different categories. The next table provides a cardinal additively-separable representation:

(v_i^l, v_i^r)	*Agent 1*	*Agent 2*	*Agent 3*	*Agent 4*
(l_1, r_1)	$1, 3$	$3, 5$	$0, 0$	$0, 0$
(l_2, r_2)	$0, 2$	$5, 1$	$0, 3$	$2, 0$
(l_3, r_3)	$0, 0$	$0, 2$	$5, 1$	$5, 3$
(l_4, r_4)	$3, 0$	$0, 0$	$2, 5$	$1, 5$

For instance, agent 1*'s ordinal preference* $(l_4, r_1) \succ_1 (l_4, r_2)$ *is represented by* $v_1^l(l_4) + v_1^r(r_1) = 3 + 3 > v_1^l(l_4) + v_1^r(r_2) = 3 + 2$.

3.3 Gains from trade

We end the section with a third sufficient condition that captures the requirement that larger coalitions can, by trading, achieve at least the utilities that smaller constituent coalitions can achieve.

An economy satisfies *gains from trade* if, for any two coalitions S and S', any S-allocation $\{X_i : i \in S\}$ and any S'-allocation $\{X'_i : i \in S'\}$, there exists an $S \cup S'$-allocation $\{Y_i : i \in S \cup S'\}$ with $v_h(Y_h) \geq \min\{v_h(X_h), v_h(X'_h)\}$ for all $h \in S \cup S'$; where we define $X'_h = X_h$ when $h \in S' \setminus S$ or $h \in S' \setminus S$.

Theorem 3. *If an economy has gains from trade, and all agents' utilities are injective, then the weak core is nonempty.*

The proof of Theorem 3 is in Section 5. Like the other results in this section, the proof proceeds by showing that Scarf's sufficient condition is satisfied. It is, however, of some interest that another proof is possible by showing that an economy as given in the theorem is associated to an ordinally convex NTU game. These are known to have non-empty weak core (Peleg and Sudhölter (2007, Theorem 12.3.3)). This alternative proof is also included in Section 5.

The assumption of gains from trade is violated in our Example 2. Indeed consider the coalitions $S = \{1, 2\}$ and $S' = \{2, 3\}$, with the S-allocation $(X_1, X_2) = ((\ell_1, r_2), (\ell_2, r_1))$, and the S'-allocation $(X'_2, X'_3) = ((\ell_2, r_3), (\ell_3, r_2))$. It is clear that any allocation in $S \cup S'$ that gives 1 at least the utility from X_1 must give 1 the bundle X_1. This rules out that 3 gets r_2, so 3 would have to receive (ℓ_1, r_3) in order to be as well off as in X_3'. This is, however incompatible with 1 getting ℓ_1.

4 A generalization of the TTC algorithm

We propose an algorithm for the problem of finding stable allocations in discrete economies. It generalizes Gale's Top Trading Cycles (TTC) for the unit-demand housing model, in the sense that when applied in a housing economy it will essentially replicate the behavior of the TTC (and, of course, find a weak core allocation). We present some sufficient conditions under which our algorithm is guaranteed to find a stable outcome.

Suppose that v_i is injective, and let $U_i = v_i(2^O)$ be the set of all possible values that i's utility can take. Note that U_i is a finite set with cardinality 2^O; in fact each utility value $u_i \in U_i$ may be identified with a bundle $v_i^{-1}(u_i) \subseteq O$.

We call a collection $X = \{X_i : i \in A\}$ with $\cup_i X_i \subseteq O$ a *preallocation.* Note a pairwise-disjoint preallocation is an allocation. A preallocation X is *individually rational* if $v_i(X_i) \geq v_i(\omega_i)$ for all i. A profile of utilities $u = (u_1, \ldots, u_n) \in U = \times_i U_i$ is identified with a preallocation.

Given $u \in U$ and $k = 2, 3, \ldots, n$, let

$$B_i^k(u) = \{u_i' : \exists S\text{-allocation } X \text{ st } |S| \leq k, i \in S, v_i(X_i) = u_i' \\ \text{and for all } j \in S \setminus \{i\}; v_j(X_j) \geq u_j\}.$$

$B_i^k(u)$ denotes the set of utilities that i can achieve through a trade in a coalition of size at most k, and that assures each of i's trading partner j at least the utility u_j. Since i can form a coalition $\{i\}$, $v_i(\omega_i) \in B_i^k(u)$.

Now define a function $T_k : U \to U$ by $(T_k u)_i = \max B_i^k(u)$. We use the notation $T_k u$ rather than $T_k(u)$ for the values of the function T_k. In words, the function $T_k u$ lets each agent i obtain the best possible utility that they can achieve by forming a trade with at most k agents, and by ensuring that each of her trading partners enjoys at least the utility that they are guaranteed to obtain in the vector u. Importantly, the vector u corresponds to a *pre*-allocation: it could be a "fantasy" in which multiple agents consume the same goods; and the image $T_k u$. We shall see in Lemma 2, however, that when utilities are monotonic then the fixed points of T_k correspond to actual allocations.

In the following, we will work with either $k = 2$ of $k = |A|$. In each case the value of k will be obvious and so we shall suppress the dependence of T on k in our notation. The algorithm we are interested in consists of iterates of the function T, and so we shall denote by T^m the composition of T with itself m times. Formally, the mth iterate of T is defined recursively by $T^m u = T(T^{m-1}u)$ for $m = 2, 3, \dots$. The key observation is that the function T is monotone decreasing, so that $u \leq u'$ implies that $Tu' \leq Tu$. In turn, this means that the composition of T with itself, T^2, is monotone increasing. Tarski's fixed point theorem therefore applies to the mapping $u \mapsto T^2 u$.

4.1 Gale's Top Trading Cycles algorithm in the Shapley-Scarf housing model

When specialized to the housing market of Shapley and Scarf (1974), our algorithm replicates the TTC. In fact it can be seen to mimic the TTC, iteration by iteration.

Consider a Shapley-Scarf housing market $\{A, O = (h_i)_{i\in A}, (\succ_i)_{i\in A}\}$, and a version of the TTC algorithm such that each round clears all cycles. Then the TTC defines a partition of the set of agents, $A_1, A_2, \dots$, in which each A_r consists of the set of agents who get the final allocated houses in round r of the algorithm.

Let v_i be a utility representation of i's preferences $\succ_i$. Denote by T the function $T_n : U \to U$, as defined above. In words, we allow for coalitions of any size ($k = |A|$) in defining the maximal utility that an agent may achieve in a trade.

Now we relate the TTC to the sequence of iterations of T starting from the vector $\underline{u} \in U$: $T^m \underline{u}$, $m = 1, 2, \dots$, where $\underline{u} = (v_i(h_i))_{i\in A}$. That is, we start iterating T from the allocation in which each agent is consuming their own house.

Observe that $\underline{u} \leq T\underline{u}$. Since T is decreasing and $\underline{u}$ is the minimum individually rational pre-allocation, $\underline{u} \leq T^2\underline{u} \leq T\underline{u}$. Then, as T is decreasing, $T^2\underline{u} \leq T^3\underline{u} \leq T\underline{u}$. Subsequently,

$$\underline{u} \leq T^2\underline{u} \leq \cdots \leq T^3\underline{u} \leq T\underline{u}. \tag{1}$$

Recall that we index the iterations of TTC by r, and that $A_r \subseteq A$ is the set of agents who obtain their final house in the rth iteration of TTC. The next result states that the *even* iterations of T mimic the different rounds of the TTC algorithm.

Lemma 1. $\forall r = 1, 2, \ldots$ *and* $i \in A_r$,

$$T^{2r-1}\underline{u}_i = T^{2r}\underline{u}_i = T^{2r+1}\underline{u}_i = \ldots .$$

4.2 Fixed points of T and T^2

The formulation of the function T is inspired by similar constructions in the literature on two-sided matchings.[5] There the two-sided nature of the problem allows for the existence of a fixed point of T, under standard assumptions on preferences. Trading in our model is *not two sided*, so there is no hope of replicating the ideas in the two-sided models. Instead, we use an idea exploited by Echenique and Yenmez (2007) to work with the fixed points of T^2.

The following result establishes the basic properties of the fixed points of T and of T^2.

Lemma 2. *Suppose that v_i is strictly monotone and injective. Consider the function $T : U \to U$ as defined above for some k.*

- *There exists $u \in U$ such that $u = T^2 u$, $u \leq Tu$, and the preallocation defined by u is individually rational.*
- *If $u = Tu$ the preallocation defined by u, namely $\{v_i^{-1}(u_i) : i \in A\}$, is an allocation that is individually rational and not blocked by any coalition of size at most k.*

Now it is possible to find all fixed points of T^2, which are guaranteed to contain all fixed points of T, if they exist. We focus instead in the next section on sufficient assumptions that guarantee that a fixed point of T^2 (which exists by Tarski's fixed point theorem) delivers an allocation in the bargaining set. This solution has some algorithmic implication as well, as we remark after stating Theorem 4 below.

4.3 Discrete TU economies

A pair of agents i and j *objects* to an allocation X if there exists an $\{i, j\}$-allocation (Y_i, Y_j) with $v_i(X_i) \leq v_i(Y_i)$, $v_j(X_j) \leq v_j(Y_j)$, and at least one

[5] See, for example, Adachi (2000), Echeniquea and Oviedo (2004), Echenique and Oviedo (2006), Fleiner (2003), Hatfield and Milgrom (2005), and Ostrovsky (2008). An early precedent for this idea, and for looking at T^2, is in Roth (1975).

inequality being strict. We say that the $\{i,j\}$-allocation (Y_i, Y_j) is a *pairwise objection* to X.

A pairwise objection defines an allocation $\bar{X}$ by setting $\bar{X}_h = Y_h$ for the objecting pair of agents; $\bar{X}_h = X_h$ for all agents for whom X_h is obtained through an S-allocation, with $i, j \notin S$; and $\bar{X}_h = \omega_h$ for those agents who were receiving X_h through an S-allocation with S being a coalition that contains one (at least) of the objecting agents.

A pair of agents i' and j' *counterobjects* the pairwise objection (Y_i, Y_j) if there is a $\{i', j'\}$-allocation $(Y'_{i'}, Y'_{j'})$ so that:

- $|\{i,j\} \cap \{i',j'\}| = 1$, and
- $(Y'_{i'}, Y'_{j'})$ is a pairwise objection to $\bar{X}$.

The set of allocations for which there is no pairwise objection without a pairwise counterobjection is termed the *pairwise bargaining set.*

The pairwise bargaining set is adapted from the bargaining set; a solution concept proposed for general cooperative games, and studied extensively in general equilibrium theory. The bargaining set was first proposed by Aumann and Maschler (1964). See Mas-Colell (1989b) and Zhou (1994) for alternative versions. For applications in two-sided matching, see, for example, Klijn and Massó (2003) and Echenique and Oviedo (2006). Peleg (1963) provides an existence result under the assumption that only pairs of agents can generate value, which may operationally render the bargaining set equivalent to ours, but he works with convex-valued, in clear contradiction with the assumptions of a discrete economy.[6]

We prove the non-emptyness of the pairwise bargaining set under two additional assumptions on agents' utility functions. The first such assumption is strict monotonicity. The second is termed "discrete transferable utility." Importantly, the technique for showing the existence of an allocation in the pairwise bargaining set is algorithmic. It results from considering fixed points of the function T we have introduced above, under the restriction of $k = 2$; that is, we focus on coalitions of size 2.

To describe the assumption needed on agents utility, we first introduce the set of utilities that a pair of agents can achieve through bargaining. For each pair of agents, i and j, let the function $v_{i,j}$ describe the Pareto frontier between them, from the view-point of agent i. That is, $v_{i,j}(\theta)$ is the value of

[6]Peleg's argument relies on a topological fixed-point theorem. It is not constructive.

the problem $P_{i,j}$ defined by:

$$\begin{array}{cc} \max_{(X_i,X_j)} & v_i(X_i) \\ \text{s.t.} & \begin{cases} X_i \cup X_j \subseteq \omega_i \cup \omega_j \\ v_j(X_j) \geq \theta. \end{cases} \end{array}$$

When $\theta \in \mathbf{R}$ is such that the above problem has a solution for some $\theta' \geq \theta$, we say that it is *feasible.* When θ is not feasible we define $v_{i,j}(\theta) = -\infty$. Observe that the function $v_{i,j}$ is monotone decreasing, and describes the Pareto frontier between agents i and j.

Say that the economy satisfies *discrete transferable utility* if, for all θ, θ', with $\theta' > \theta$ and θ' being feasible, the following inequality holds:[7]

$$v_{i,j}(\theta) - v_{i,j}(\theta') \leq \theta' - \theta.$$

Theorem 4. *Let $E = (O, \{(v_i, \omega_i) : i \in A\})$ be an economy in which each v_i is strictly monotone and injective. If E satisfies discrete transferable utility, then it has an allocation that is individually rational and in the pairwise bargaining set.*

Remark. An algorithm exists that, with oracle access to T, finds an allocation in the bargaining set in time $O(\log^{|A|} M)$, where $M = \max\{|\{X_i : v_i(X_i) \geq v_i(\omega_i)\}| : i \in A\}$. This algorithm, which combines the standard idea of iterating the monotone increasing function $u \mapsto T^2 u$ on the lattice U with binary search, is easily formulated given the recent literature on finding a Tarski fixed point: see, for example, Etessami et al. (2020).[8]

Ehlers (2007) and Ehlers and Morrill (2020) consider versions of the von-Neumann-Morgenstern stable sets in assignment problems. Stable sets are also an area of interest for the discrete economies that we consider in our paper. We argue, using the example in Section 2, that the stable set can be empty. So one would have to impose additional assumptions in order to

[7]When v_i and v_j are strictly monotone, the condition implies a slope of -1 of the Pareto frontier, which motivates the name "discrete transferable utility." If (X_i, X_j) and (Y_i, Y_j) are distinct Pareto optimal $\{i, j\}$-allocations and $v_j(X_j) < v_j(Y_j)$, then the condition implies $v_i(X_i) - v_i(Y_i) \leq v_j(Y_j) - v_j(X_j)$, and applying the condition for j yields $v_j(Y_j) - v_j(X_j) \leq v_i(X_i) - v_i(Y_i)$. Taken together, $v_i(X_i) + v_j(X_j) = v_i(Y_i) + v_j(Y_j)$.

[8]The recent literature on finding a Tarski fixed point includes Dang et al. (2020) (in unpublished work that was perhaps the first in proposing the combination of iteration and binary search), Chang et al. (2008), Dang and Ye (2018), and Fearnley and Savani (2021).

obtain existence. Indeed if we consider the example, and we focus on individually rational allocations, we see that the endowments and three alternative allocations (X, Y, Z) should be considered. The three allocations are cyclically blocked by each other: X is blocked by Y, which is blocked by Z, and in turn by X. If a stable set includes allocation 1, then it must include the allocation 3. This, however, violates internal stability.

5 Proofs

We prove Theorem 1 and Theorem 2 by showing that the games satisfy the sufficient condition of Scarf (1967) for non-emptiness of weak core in NTU games. Before presenting our proofs, we formally recall the definition of a NTU game and the result of Scarf.

A *NTU game* is a pair (A, V) where A is set of agents and V is a function that associates with each coalition $S \subset A$ a subset $V(S) \subset \mathbf{R}^A$ such that

- $V(S)$ is closed
- if $x \in V(S)$ and $y \in \mathbf{R}^A$ with $y_i \leq x_i$ for all $i \in S$ then $y \in V(S)$
- the set $[V(S) \setminus \cup_{i \in S} intV(\{i\})] \cap \mathbf{R}^S \times \{0\}^{A \setminus S}$ is non-empty and bounded

Now to state Scarf's result, we first define the notion of a balanced collection of coalitions and a balanced game.

Given a NTU game (A, V), a family of subsets $\mathcal{S}$ is said to be a *balanced collection of coalitions* if there exist non-negative weights $\delta_S, S \in \mathcal{S}$ such that

$$\sum_{\{S \in \mathcal{S}: S \ni i\}} \delta_S = 1 \text{ for all } i \in A.$$

A NTU game (A, V) is *balanced* if for every balanced collection of coalitions $\mathcal{S}$ and $u \in \mathbf{R}^A$,

$$u \in \cap_{S \in \mathcal{S}} V(S) \implies u \in V(A)$$

Lemma 3 (Scarf (1967))**.** *A balanced NTU game (A, V) always has a non-empty weak core.*

Scarf's lemma is included without proof. We use it to prove Theorem 1 and 2. These proofs are based on different rounding constructions, which is also how Shapley and Scarf (1974) establish Scarf's sufficient condition for the housing model. The situation is, however, quite a bit more involved; especially in the case of Theorem 2.

5.1 Proof of Theorem 1

Given any economy $E = \{O, \{(v_i, \omega_i) : i \in A\}$ with dichotomous item preferences, consider the NTU game (A, V) where $V(S)$ is defined to be the set of all $u \in \mathbf{R}^A$ for which there exists an S-allocation $\{X_i : i \in S\}$ with $u_i \leq v_i(X_i)$ for $i \in S$. We will show that the NTU game (A, V) is balanced.

Let $\mathcal{S}$ be a balanced collection of coalitions with weights $(\delta_S)_{S\in\mathcal{S}}$ so that $\sum_{\{S\in\mathcal{S}:S\ni i\}} \delta_S = 1$.

Suppose that $u \in \cap_{S\in\mathcal{S}} V(S)$. For each u_i, let t_i be the smallest integer that is greater than or equal to u_i. Since utilities are always integral in this economy, we know that for each $S \in \mathcal{S}$ there is an S-allocation $\{X_i : i \in S\}$ such that t_i is the number of good items received by agent i.

The integers t_i are called *target utilities*. Without loss of generality, we assume that $t_i \geq 1$ (the case when $t_i = 0$ may be dealt with trivially). For each $S \in \mathcal{S}$, let P_S be a $|A| \times |O|$ matrix describing the corresponding S-allocation $\{X_i : i \in S\}$ so that $P_S(i, j) = 1$ if $o_j \in X_i$ and 0 otherwise.

Let

$$P = \sum_{S\in\mathcal{S}} \delta_S P_S$$

Observe that P has two properties:

1. Row i sums up to at least $t_i \geq 1$.

2. Each column sums up to at most 1.

Property (1) holds because each P_S has at least t_i ones in row i and $\sum_{S\ni i} \delta_S = 1$. Property (2) holds because for each o and each S, there is a 1 in column o iff o's owner (say i) is a member of S and o is assigned in the S-allocation. Since (again) $\sum_{S\ni i} \delta_S = 1$, property (2) follows.

Let B be any $|A| \times |O|$ matrix that satisfies these two properties. Say that an entry $b_{i,j}$ is fractional if $b_{i,j} \in (0, 1)$.

Suppose that there is a row i that has less than t_i entries that are ones. Since row i adds up to at least t_i, and no entry can be larger than one, there must exist a column j for which the entry $b_{i,j} \in (0, 1)$. Now there are two possibilities. Either $b_{i,j}$ is the only non-zero entry in column j, or there is another row h with $b_{h,j} > 0$. If the former case is true, we can increase $b_{i,j}$ to one and still obtain a matrix that satisfies the two properties but has strictly fewer fractional entries.

Suppose the latter case is true. Since $b_{i,j} > 0$ and B satisfies property (2), $b_{h,j} < 1$. Now there are again two cases. Either there are t_h ones in row h, or not. If there are already t_h ones in row h then since $b_{h,j} < 1$ the columns with ones cannot include j. So we can decrease $b_{h,j}$ without affecting the property that the entries of row h sum up to at least t_h. Now find the largest $\varepsilon > 0$ such that $b_{i,j} + \varepsilon \leq 1$ and $b_{h,j} - \varepsilon \geq 0$. Now replace $b_{i,j}$ with $b_{i,j} + \varepsilon$ and $b_{h,j}$ with $b_{h,j} - \varepsilon \geq 0$. We obtain a new matrix that satisfies the two properties but has strictly fewer fractional entries.

Finally, suppose that there are not t_h ones in row h. Given that row h adds up to at least t_h there must exist another fractional entry $b_{h,l} \in (0, 1)$. Starting from $b_{h,l} \in (0, 1)$, repeat the argument above.

This procedure will either result in a new matrix that satisfies the two properties and has strictly fewer fractional entries, or we will obtain a cycle $(i_1, j_1), \ldots, (i_M, j_M) = (i_1, j_1)$ where

- $b_{i_m,j_m} \in (0, 1)$ is fractional;
- each odd-numbered entries are in the same row as the next entry, and in the same column as the preceding one.

Now we can add $\varepsilon > 0$ to each odd-numbered entry, and subtract $-\varepsilon$ from the next entry. This will keep the same row and column sums. Choose the largest ε that ensures that each entry is in $[0, 1]$. This again results in a new matrix that satisfies properties (1) and (2) and has strictly fewer fractional entries.

The argument we laid out implies that for any matrix satisfying (1) and (2), as long as there is some row that does not have t_i ones, there is a new matrix that satisfies properties (1) and (2) and has strictly fewer fractional entries. So there must then exist a matrix satisfying the two properties and where all rows have at least t_i ones.

This matrix describes an allocation in the economy in which each agent i gets utility at least $t_i \geq u_i$. Therefore Scarf's balancedness condition is satisfied, and the weak core is nonempty.

5.2 Proof of Theorem 2

Similar to the proof of Theorem 1, we show that the NTU game (A, V) defined by a categorical economy is balanced. Let $\mathcal{S}$ be a balanced collection of coalitions with weights $(\delta_S)_{S \in \mathcal{S}}$, and take $u \in \cap_{S \in \mathcal{S}} V(S)$. For each u_i, let

t_i be the smallest integer that is greater than or equal to u_i. In the proof, we refer to t_i as the *target utility* for agent i.

Note that $t_i \in V_i(S)$ for all S. That means each $S \in \mathcal{S}$ has a S-allocation $\{X_i : i \in S\}$ such that t_i is the number of categories in which agent i gets a good item. We can wlog focus on the case when $t_i \geq 1$. Also, let P_S be a zero-or-one matrix with $|A|$ rows and $\sum_k |O^k|$ columns. An entry at (i, j) is one iff object j is in X_i. Define $P = \sum_{S\in\mathcal{S}} \delta_S P_S$.

We define an algorithm that manipulates the matrix P and ultimately produces an (integer) allocation that achieves the target utilities $t = (t_i)_{i\in A}$. This will prove that the game (A, V) is balanced, so the nonempty weak core by Scarf (1967).

The algorithm operates over a matrix B, augmented by a column vector $\tilde{t}$. Initially, $(B|\tilde{t}) = (P|t)$ and satisfies:

1. All entries of B are in $[0, 1]$.
2. Each column sums up to 1.
3. If $b_{i,j} > 0$, then $v_i^k(j) = 1$ for the category k of object that j is.
4. $\sum_k \sum_{j\in O^k} b_{i,j} \geq \tilde{t}_i$; and for each k, $\sum_{j\in O^k} b_{i,j} \leq 1$.

The algorithm is composed of two major subroutines: preprocessing and rounding.

1. **Preprocessing**

   We apply the following procedures in no particular order until there is no further update to $(B|\tilde{t})$. The number of rows may increase, so we refer to i as a row rather than an agent.

   (a) For row i and category k such that $\sum_{j\in O^k} b_{i,j} = 1$ (a guaranteed utility of 1 from category k) and $\tilde{t}_i > 1$ (additional guaranteed utility), divide row i into two rows. One row inherits category k ($\{b_{i,j}\}_{j\in O^k}$), the other row inherits other categories ($\{b_{i,j}\}_{j\notin O^k}$), and the remaining entries of the two rows are zero. Accordingly, expand the target-utility vector $\tilde{t}$ such that $\tilde{t}_i$ is divided into two elements 1 and $\tilde{t}_i - 1$.

   (b) Remove any row i with $\tilde{t}_i = 0$ (row i gets no object), and any column j with $\sum_i b_{ij} = 0$ (no row gets j).

(c) For j, if there exists a unique i such that $b_{i,j} > 0$, then increase $b_{i,j}$ to 1, and reduce $b_{i,j'} = 0$ for any j' in the same category as j.

(d) Replace any entry $b_{i,j} = 1$ with 0 and reduce $\tilde{t}_i$ by 1 (j is assigned to the agent associated with row i).

2. **Rounding**

   $(B|\tilde{t})$ satisfies:

   (a) All entries of B are in $[0, 1)$.

   (b) for each j, $0 < \sum_i b_{i,j} \leq 1$ and there exist at least two rows i, i' with $b_{i,j} > 0$ and $b_{i',j} > 0$.

   (c) $\sum_k \sum_{j \in O^k} b_{i,j} \geq \tilde{t}_i \in \{1, 2, \ldots, k-1\}$; and $\max_k \sum_{j \in O^k} b_{i,j} \leq 1$.[9]

   Here we consider a graph (V, E) such that V is the set of non-zero elements of B and E is the set of pairs of vertices that are in same row or column.

   We claim that there exists a cycle. Take any (i, j) with $b_{i,j} > 0$. By property 2b, we find $i' \neq i$ such that $b_{i',j} > 0$. Since the guaranteed utility $\tilde{t}_{i'} \geq 1$ (property 2c), there exists $j' \neq j$ such that $b_{i',j'} > 0$. By property 2b, we subsequently find $i'' \neq i'$ such that $b_{i'',j'} > 0$. When some agent appears twice in this construction of a path, a cycle $(i_1, j_1), \ldots, (i_M, j_M) = (i_1, j_1)$ is found. Label the entries in the cycle such that each odd-numbered entries are in the same row as the next entry, and in the same column as the preceding one.

   We add $\epsilon > 0$ for each odd-numbered entry, and subtract ϵ from the next entry. The row and column sums remain the same. Choose the largest ϵ such that either (i) one entry in the cycle becomes integral, i.e., 0 or 1, or (ii) there exists a row i and category k such that the constraint $\sum_{j \in O^k} b_{i,j} v_i^k(j) \leq 1$ has become newly binding.

We iterate the preprocessing and rounding subroutines. Preprocessing weakly decreases the number of fractional elements of B, even if the size of the matrix may increase. Rounding either (i) decreases the number of fractional elements by at least one, or (ii) creates a new pair, consisting of

[9] $\tilde{t}_i \neq 0$ because we applied preprocessing 1b; $\tilde{t}_i \neq k$ because we applied preprocessing 1a.

a row i and a category k, such that $\sum_{j \in O^k} b_{i,j} = 1$; and such pair forms a new row of $(B|\tilde{t})$ in the follow-up preprocessing step. Hence, the algorithm terminates in at most $|P| + (|A| \times (K-1))$ rounds. When it terminates, the matrix B is integral.

Finally we identify an allocation from the integral matrix B and the assignments by preprocessing 1d given to agents associated with the rows. The target utility t is achieved because of property 2c of the matrix B and the sum of elements of $\tilde{t}$ associated with each agent has decreased by one only upon an assignment a good (acceptable) item by preprocessing 1d. Each agent consumes at most one good item of each category k because, when the algorithm runs, at most one row i associated with her has $\sum_{j \in O^k} b_{i,j} > 0$, until a possible assignment of an object j in category k by preprocessing 1d, after which every row i associated with the agent has $\sum_{j \in O^k} b_{i,j} = 0$.

The sufficient condition in Scarf's lemma is satisfied, and the proof is done, as we have produces an allocation in which all agents achieve their target utilities.

5.3 Proof of Theorem 3

Given an economy $E = (O, \{(v_i, \omega_i) : i \in A\})$, we define a NTU game (A, V) by letting $V(S)$ for non-empty $S \subseteq A$ be the set of all $u \in \mathbf{R}^A$ for which there exists an S-allocation $\{X_i : i \in S\}$ with $u_i \leq v_i(X_i)$ for $i \in S$. Lastly, we let $V(\emptyset) = \{0\}$.

We present two proofs. The first proof shows that, if an economy has gains from trade, the induced NTU game (A, V) is balanced and therefore, has a non-empty weak core. Indeed suppose that B is a balanced collection of coalitions and $u \in \cap_{S \in B} V(S)$. We want to show that $u \in V(A)$. Take any $S, S' \in B$. We know there exists S allocation X and S' allocation X' such that $u_i \leq v_i(X_i)$ for $i \in S$ and $u_i \leq v_i(X'_i)$ for $i \in S'$. By gains from trade, there exists $S \cup S'$ allocation Y such that $v_h(Y_h) \geq \min\{v_h(X_h), v_h(X'_h)\} \geq u_i$ for $h \in S \cup S'$. We can now apply gains from trade to coalitions $T = S \cup S'$ and $T' \in B$ to get a $T \cup T'$ allocation in which every agent $i \in T \cup T'$ gets at least u_i. Eventually, we'll get an allocation in which each agent $i \in A$ gets at least u_i and so we get that $u \in V(A)$.

Next, we present a second proof, based on the theory of ordinally convex games: The defined game is ordinally convex if for all $S, S' \subseteq A$,

$$V(S) \cap V(S') \subseteq V(S \cap S') \cup V(S \cup S'). \tag{2}$$

An ordinaly convex game has non-empty weak core (Peleg and Sudhölter (2007, Theorem 12.3.3)). We proceed to show that the game we have defined is ordinally convex. Consider $S, S' \subseteq A$. If $S \subseteq S'$, then (2) holds trivially because $V(S) = V(S \cap S')$ and $V(S') = V(S \cup S')$. A similar conclusion holds if $S' \subseteq S$. Thus, we suppose that $S \cap S'$ is a strict super set of S and S'.

Take $u \in V(S) \cap V(S')$. There are S-allocation $\{Y_i : i \in S\}$ and S'-allocation $\{Y'_i : i \in S'\}$ such that $u_i \leq v_i(Y_i)$ for $i \in S$ and $u_j \leq v_j(Y'_j)$ for $j \in S'$. If S and S' are disjoint, then $\{Y_i : i \in S\} \cup \{Y'_j : j \in S'\}$ is an $S \cup S'$-allocation. Then, $u \in V(S \cup S')$. Consider the other case of $S \cap S' \neq \emptyset$. If $Y_i = Y'_i$ for $i \in S \cap S'$, then by injective utilities, no object in $\{Y_i : i \in S \cap S'\}$ $(= \{Y'_i : i \in S \cap S'\})$ is from the endowments of $S \backslash S'$ or $S' \backslash S$. Thus, $\{Y_i : i \in S \cap S'\}$ is an $S \cap S'$-allocation. It follows that $u \in V(S \cap S')$. On the other hand, if for some $j \in S \cap S'$, $v_j(Y_j) < v_j(Y'_j)$ (we omit the other case of $v_j(Y_j) > v_j(Y'_j)$), then gains from trade implies that there exists an $S \cup S'$- allocation $\{Z_i : i \in S \cup S'\}$ with $v_i(Z_i) \geq \min\{v_i(Y_i), v_i(Y'_i)\}$ for all $i \in S \cup S'$. Hence, $u_i \leq v_i(Z_i)$ for all $i \in S \cup S'$, which implies $u \in V(S \cup S')$.

Remark. A similar proof approach is not applicable for an economy with dichotomous preferences or a categorical economy, because they may not define an ordinally convex game. For example, consider an economy with three agents $\{1, 2, 3\}$ such that agents 1 and 2 consider each others' endowment acceptable, and agents 2 and 3 consider each other's endowment acceptable. For any coalition $\{i, j\}$ with $(i = 1, j = 2)$ or $(i = 2, j = 3)$, $(1, 1, 1) \in V(\{i, j\})$ because there exists a $\{i, j\}$-allocation $\{X_i = \omega_j, X_j = \omega_i\}$ such that $1 \leq u_i(X_i)$ and $1 \leq u_j(X_j)$. However, $(1, 1, 1) \neq V(\{1, 2, 3\})$ because agents 1 and 3 consider only agent 2's endowment acceptable. The convexity condition (2) does not hold.

6 Lemmas regarding the T algorithm

6.1 Proof of Lemma 1

We prove the lemma by induction.

- $(r = 1)$ The utility $T\underline{u}$ identifies an A_1-allocation. Formally, $\{v_i^{-1}((T\underline{u})_i) : i \in A_1\}$ is an A_1-allocation such that each $i \in A_1$ gets her most preferred house. Consequently, for $i \in A_1$, $T\underline{u}_i = T^2\underline{u}_i$, and (1) implies that $(T\underline{u})_i = (T^2\underline{u})_i = (T^3\underline{u})_i = \dots$.

- ($r = 2$) Given $T\underline{u}$, no agent in A_1 is willing to trade with an agent not in A_1. Hence, for $i \notin A_1$, $(T^2\underline{u})_i$ is a house that is not allocated in the first round of TTC. Thus, $T^3\underline{u}$ identifies an A_2-allocation such that an agent in A_2 gets her most preferred remaining house after the first round of TTC. Consequently, for $i \in A_2$, $(T^3\underline{u})_i = (T^4\underline{u})_i$, and (1) implies that $(T^3\underline{u})_i = (T^4\underline{u})_i = (T^5\underline{u})_i = \dots$.

- ($r > 2$) A proof is similar to the previous step, so we omit.

6.2 Proof of Lemma 2: Part 1

If $u \leq u'$ then $B_i^k(u') \subseteq B_i^k(u)$, which implies that $Tu' \leq Tu$. In turn this means that T^2 is monotone increasing. Let $\underline{u} = (v_i(\omega_i))_{i \in A}$ and note that $\underline{u} \leq Tu$ for all u. In particular, $\underline{u} \leq T^2(\underline{u})$, which implies that the sequence $T^{2m}(\underline{u})$ is monotone increasing. Since U is finite, there is m so that $u = T^{2(m+1)}(\underline{u}) = T^{2m}(\underline{u})$. Such u is a fixed point of T^2. The preallocation defined by u is individually rational because $u = T^2(u) \geq \underline{u}$.

Note $T^0\underline{u} = \underline{u} \leq T\underline{u} = T^1\underline{u}$, and T^2 is monotone increasing. Thus, $T^{2m}\underline{u} \leq T^{2m+1}\underline{u}$, and we have $u \leq Tu$ for the above fixed point u.

6.3 Proof of Lemma 2: Part 2

We prove an auxiliary lemma first.

Lemma 4. *Take u such that $T^2(u) = u$ and $u \leq T(u)$, and partition the set of agents such that $A_1 \equiv \{i : u_i = (Tu)_i\}$ and $A_2 \equiv \{i : u_i < (Tu)_i\}$. Then, the preallocation defined by u ($\{v_i^{-1}(u_i) : i \in A\}$) is an A_1-allocation: (1) $\cup_{i \in A_1} v_i^{-1}(u_i) = \cup_{i \in A_1} w_i$ and (2) $v_i^{-1}(u_i) \cap v_j^{-1}(u_j) = \emptyset$ for $i, j \in A_1$.*

Proof. For $i \in A_1$, $(T^2u)_i = u_i = (Tu)_i$ implies that there exists a coalition S^i with $i \in S^i$, $|S^i| \leq k$, and an S^i-allocation $\{Y_j : j \in S^i\}$ such that $v_i(Y_i) = u_i$ and $v_j(Y_j) \geq (Tu)_j$ for $j \in S^i \backslash \{i\}$. This S^i-allocation ensures $(Tu)_j$ for all $j \in S^i$; so $u_j = (T(Tu))_j \geq v_j(Y_j) \geq (Tu)_j$. On the other hand, $u \leq Tu$, so $u_j = v_j(Y_j) = (Tu)_j$, and $S^i \subseteq A_1$. By injectivity of utility, $Y_j = v^{-1}(u_j)$ and $\{v_j^{-1}(u_j) : j \in S^i\}$ is an S^i-allocation.

Let $\mathcal{S} = \{S^i : i \in A\}$. Maybe $S^i = S^j$ for some $i, j \in A_1$.

We use $\mathcal{S}$ to construct a partition P of A_1. Let $X_i = v_i^{-1}(u_i)$ for $i \in A_1$. Take any $S, S' \in \mathcal{S}$ and note that $\{X_i : i \in S\}$ is an S-allocation, and $\{X_j : j \in S'\}$ is an S'-allocation. Clearly, $\cup_{h \in S \cap S'} X_h \subseteq \cup_{h \in S \cap S'} \omega_h$, so $\{X_i :$

$i \in S \cap S'\}$ is a $S \cap S'$-allocation. In fact, $\cup_{h \in S \cap S'} X_h = \cup_{h \in S \cap S'} \omega_h$, because for each $i \in S \cap S'$, $v_i(X_i) = (Tu)_i = \max B_i^k(u)$, and by monotonicity no object in $\cup_{j \in S \cap S'} \omega_j$ is left out in the $S \cap S'$-allocation $\{X_i : i \in S \cap S'\}$. Then, it follows from $\cup_{i \in S} X_i \subseteq \cup_{i \in S} \omega_i$ that $\cup_{i \in S \setminus S'} X_i \subseteq \cup_{i \in S \setminus S'} \omega_i$, i.e., $\{X_i : i \in S \setminus S'\}$ is an $S \setminus S'$-allocation. Similarly, $\{X_i : i \in S' \setminus S\}$ is $S' \setminus S$-allocation. We substitute S and S' in $\mathcal{S}$ with $S \cap S'$, $S \setminus S'$, and $S' \setminus S$.

A recursive substitutions results in a partition P of A_1 such that for $S \in P$, $\{v_i^{-1}(u_i) : i \in S\}$ is a S-allocation. Then, $v_i^{-1}(u_i) \cap v_j^{-1}(u_j) = \emptyset$ for distinct $i, j \in S$, and also for $i \in S$ and $j \notin S$, which implies that $\{v_i^{-1}(u_i) : i \in A_1\}$ is an A_1-allocation. □

For Part 2 of Lemma 2, note that $u = Tu$ implies $A_1 = A$. Lemma 4 implies that the preallocation defined by u is an allocation.

The allocation defined by u is individually rational because any point in the image of T is individually rational. Similarly, if there is a coalition S of size at most k and an S-allocation X such that $v_i(X_i) \geq u_i$ for $i \in S$ with some inequality strict, then this would violate $u_i = (Tu)_i$ for $i \in S$.

6.4 Proof of Theorem 4

This proof makes use of the T-algorithm we introduced in Section 4 and the set of agents $A_2 = \{i : u_i < (Tu)_i\}$ defined in Lemma 4.

Lemma 5. *Consider T-algorithm defined for the case of $k = 2$. For any $i \in A_2$, there exists $j \in A_2 \setminus i$ uniquely, and vice versa, such that $((Tu)_i, u_j)$ is an $\{i, j\}$-allocation, i.e., $v_i^{-1}((Tu)_i)$ and $v_j^{-1}(u_j)$ partition $\omega_i \cup \omega_j$.*

Proof. Since $i \in A_2$, $u_i = (T^2u)_i < (Tu)_i$. Since $k = 2$, there exists j and an $\{i, j\}$-allocation (X_i, X_j) such that $v_i(X_i) = (Tu)_i > u_i$ and $v_j(X_j) \geq u_j$. Clearly, $j \neq i$ because $(Tu)_i > u_i \geq v_i(\omega_i)$ (some object in X_i must be from ω_j), where the last inequality holds because u is in the image of T.

In fact, $j \in A_2$. Otherwise $(Tu)_j = v_j(X_j) = u_j$. Lemma 4 showed that in the preallocation defined by u of the T-algorithm agents in A_1 trade among themselves. Since v_j is injective, there is only one bundle that guarantees j achieves utility u_j, and thus $X_j \cap \omega_i = \emptyset$. By monotonicity, then, $X_j = w_j$, a contradiction to $v_i(X_i) > u_i \geq v_i(\omega_i)$.

The uniqueness of j holds because v_i is injective. If i achieves utility $(Tu)_i$ by consuming some objects owned by j, then no other pairwise trade offer i the same utility $(Tu)_i$.

Similarly, for agent j, the uniqueness of i holds, because v_j is injective. Agent j achieves utility u_j only if she consumes an object owned by agent i. $\square$

To prove Theorem 4, we use Lemma 5 and find sequences of agents in A_2. For each sequence $(i_1, i_2, \dots, i_n)$, (i_m, i_{m+1}) for $m = 1, 2, \dots,$ and (i_n, i_1) satisfy the relation in the above Lemma.

Define a graph with A_2 as vertex-set and an edge $i \to j$ if $((Tu)_i, u_j)$ is obtained with a $\{i, j\}$ allocation. By the lemma, A_2 is partitioned into cycles. For each cycle C,

$$i_1^C \to i_2^C \cdots \to i_{|C|}^C \to i_1^C.$$

Now we claim that there are no such cycles with $|C| > 2$.

Consider an arbitrary cycle C with $|C| > 2$. Let $(i, j) = (i_m, i_{m+1})$, using summation mod $|C|$. We shall prove that:

$$(Tu)_i - u_i < (Tu)_j - u_j. \tag{3}$$

To this end, suppose first that $(Tu)_j$ is feasible in problem $P_{i,j}$. Note that $(Tu)_i = v_{i,j}(u_j)$ as $i \to j$. Moreover, $u_i > v_{i,j}((Tu)_j)$ because 1) $(Tu)_j$ is feasible in problem $P_{i,j}$ and $u_i = (T(Tu))_i$ means that $u_i \geq v_{i,j}((Tu)_j)$ and 2) the inequality is strict because v_i is injective, $i_{m-1} \to i$, and $i_{m-1} \neq j$ (as $|C| > 2$).

Thus we have that

$$(Tu)_i - u_i = v_{i,j}(u_j) - u_i < v_{i,j}(u_j) - v_{i,j}((Tu)_j) \leq (Tu)_j - u_j;$$

the last inequality following from discrete transferable utilities.

Suppose in second place that $(Tu)_j$ is not feasible in problem $P_{i,j}$. Let θ^* be the largest utility for j that is feasible: so $\theta^* = v_j(\omega_i \cup \omega_j)$ and $v_i(\emptyset) = v_{i,j}(\theta^*)$. Then we have that $u_i \geq v_i(\omega_i) > v_{i,j}(\theta^*)$ (by definition of T, and because $u = T^2 u$). Thus we again have that (3) holds because

$$(Tu)_i - u_i = v_{i,j}(u_j) - u_i < v_{i,j}(u_j) - v_{i,j}(\theta^*) \leq \theta^* - u_j < (Tu)_j - u_j,$$

the last inequality being a consequence of $(Tu)_j$ not being feasible in $P_{i,j}$.

Now (3) means that, for all m,

$$(Tu)_{i_m} - u_{i_m} < (Tu)_{i_{m+1}} - u_{i_{m+1}} \Longrightarrow \sum_{m=1}^{|C|} \left((Tu)_{i_m} - u_{i_m}\right) < \sum_{m=1}^{|C|} \left((Tu)_{i_{m+1}} - u_{i_{m+1}}\right),$$

a contradiction.

So we conclude that all cycles (if any) have length two. Partition the set A_2 into those agents who are first in their cycle (in the arbitrary enumeration of cycles we have chosen) and those who are second. Now define an allocation X by setting $X_i = v_i^{-1}(u_i)$ if either $i \in A_1$ or $i \in A_2$ and i is second in their cycle. If $i \in A_2$ and i is first then we set $X_i = v_i^{-1}((Tu)_i)$. Observe that this is indeed an allocation because, by Lemma 4, $u|_{A_1}$ defines an A_1-allocation, and for each cycle $i \to j \to i$, the utilities $(u_i, (Tu)_j)$ and $(u_j, (Tu)_i)$ are each obtained through an $\{i, j\}$-allocation.

Now suppose that a pair of agents i and j object to the allocation X. Suppose that i gets a utility that is strictly greater than in $v_i(X_i)$. Note that we cannot have $i \in A_1$, or $i \in A_2$ and first in a cycle that we used above to define the allocation X. For otherwise, in the objection, i obtains utility greater than $(Tu)_i$ by trading with j, who by objecting gets a utility $\geq u_j$; a contradiction to the definition of T. So $i \in A_2$ and must be second in the cycle. Note, finally, that in the objection i's utility must be $< (Tu)_i$ because otherwise i could achieve utility $> (Tu)_i$ while guaranteeing j a utility $\geq u_j$ (the possibility of a utility $= (Tu)_i$ being ruled out as v_i is injective).

Let h be the first agent in the cycle that i belongs to. Then h is left consuming ω_h after the objection by i and j. However, note now that there is a counter objection involving i and h by means of the allocation $(v_i^{-1}((Tu)_i), v_i^{-1}(u_h))$. This gives i utility $(Tu)_i$, which we know is $>$ than the utility i obtains in the objection. And it gives h utility $u_h \geq \omega_h$ as u is individually rational: indeed since v_h is injective we must have $u_h > \omega_h$.

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