Econ 204 2024

Lecture 1

Outline

1. Administrative Details
2. Methods of Proof
3. Equivalence Relations
4. Cardinality

Instructors

- Haluk Ergin
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- Anna Vakarova, GSI

- **Schedule:** Lectures MTWThF 9am-12noon in 534 Davis.

  Section: MTWThF 1-3:00pm in 243 Dwinelle.

  Office hours:

  Haluk: MTWThF 12noon-1pm in 517 Evans.

  Anna and Bruno: MTWThF 3-5pm, location: TBD.

- **Final Exam:** Wed August 14, 9am - 12noon, location: TBD.

- **Prerequisites:** Math 1A, 1B, 53, 54 at Berkeley or equivalent.

Course requirements:

- Problem Sets: 6 total

  (no late problem sets...no exceptions)

- Exam

Course Grade: 10% problem sets (5 highest scores out of 6), 90% final exam

Grading in First Year Economics Courses:

- median grade = B+ : solid command of material

- A and A- are very good grades, A+ for truly exceptional work

- B : ready to go on to further work...a B in 204 means you are ready to go on to 201a/b, 202a/b, 240a/b

- B- : very marginal, but we won't make you take the class again. B- in 204 means you will have a very hard time in 201a/b. Recommend you take Math 53 and 54 this year, maybe Math 104, come back next year to retake 204 and take 201a/b. B- is a passing grade, but you must maintain a B average

- C: not passing. Definitely not ready for 201a/b, 202a/b, 240a/b. Take Math 53-54 this year, maybe Math 104, retake 204 next year

- 204 with at least a B- (or a waiver from 204 requirement) is a strictly enforced prerequisite for enrollment in 201a/b

- F: means you didn't take the final exam. Be sure to withdraw if you don't or can't take the final.

Resources:

Book: de la Fuente, *Mathematical Methods and Models for Economists*

Chris Shannon's lecture notes: for every lecture + supplements for several topics

Be sure to read Corrections Handout with dlF

Seek out other references

Goals for 204

- present some particular concepts and results used in first-year economics courses 201a/b, 202a/b, 240a/b

- develop basic math skills and knowledge needed to work as a professional economist and read academic economics

- develop ability to read, evaluate and compose proofs...essential for reading and working in all branches of economics - theoretical, empirical, experimental

- **not** to review Math 53 + 54. If you are weak on this material, take Math 53-54 this year, and take 204 next year.

Learning by Doing

- to learn this sort of mathematics you need to do more than just read the book and notes and listen to lectures

- active reading: work through each line, be sure you know how to get from one line to the next

- active listening: follow each step as we work through arguments in class

- working problems: the most valuable part of the class

- you can work in groups but, always try to work through all of the problems on your own before talking to others

- best test of understanding: can you explain it to others

Methods of Proof

- Deduction
- Contraposition
- Induction
- Contradiction

We'll examine each of these in turn.

Proof by Deduction

Proof by Deduction: A list of statements, the last of which is the statement to be proven. Each statement in the list is either

- an axiom: a fundamental assumption about mathematics, or part of definition of the object under study; or
- a previously established theorem; or
- follows from previous statements in the list by a valid rule of inference

Proof by Deduction

Example: Prove that the function $f(x) = x^2$ is continuous at $x = 5$.

Recall from one-variable calculus that $f(x) = x^2$ is continuous at $x = 5$ means

$$\forall \varepsilon > 0 \ \exists \delta > 0 \text{ s.t. } |x - 5| < \delta \Rightarrow |f(x) - f(5)| < \varepsilon$$

That is, "for every $\varepsilon > 0$ there exists a $\delta > 0$ such that whenever x is within δ of 5, $f(x)$ is within ε of $f(5)$."

To prove the claim, we must systematically verify that this definition is satisfied.

Proof. Let $\varepsilon > 0$ be given. Let

$$\delta = \min\left\{1, \frac{\varepsilon}{11}\right\} > 0$$

Where did that come from ? Suppose $|x - 5| < \delta$. Since $\delta \leq 1$, $4 < x < 6$, so $9 < x + 5 < 11$ and $|x + 5| < 11$. Then

$$\begin{aligned} |f(x) - f(5)| &= |x^2 - 25| \\ &= |(x + 5)(x - 5)| \\ &= |x + 5||x - 5| \\ &< 11 \cdot \delta \\ &\leq 11 \cdot \frac{\varepsilon}{11} \\ &= \varepsilon \end{aligned}$$

Thus, we have shown that for every $\varepsilon > 0$, there exists $\delta > 0$ such that $|x - 5| < \delta \Rightarrow |f(x) - f(5)| < \varepsilon$, so f is continuous at $x = 5$. □

Proof by Contraposition

Recall some basics of logic.

$\neg P$ means "P is false."

$P \wedge Q$ means "P is true *and* Q is true."

$P \vee Q$ means "P is true *or* Q is true (or possibly both)."

$\neg P \wedge Q$ means $(\neg P) \wedge Q$; $\neg P \vee Q$ means $(\neg P) \vee Q$.

$P \Rightarrow Q$ means "whenever P is satisfied, Q is also satisfied."

Formally, $P \Rightarrow Q$ is equivalent to $\neg P \vee Q$.

Proof by Contraposition

The *contrapositive* of the statement $P \Rightarrow Q$ is the statement $\neg Q \Rightarrow \neg P$.

Theorem 1. *$P \Rightarrow Q$ is true if and only if $\neg Q \Rightarrow \neg P$ is true.*

Proof. Suppose $P \Rightarrow Q$ is true. Then either P is false, or Q is true (or possibly both). Therefore, either $\neg P$ is true, or $\neg Q$ is false (or possibly both), so $\neg(\neg Q) \vee (\neg P)$ is true, that is, $\neg Q \Rightarrow \neg P$ is true.

Conversely, suppose $\neg Q \Rightarrow \neg P$ is true. Then either $\neg Q$ is false, or $\neg P$ is true (or possibly both), so either Q is true, or P is false (or possibly both), so $\neg P \vee Q$ is true, so $P \Rightarrow Q$ is true. $\square$

Proof by Induction

We illustrate with an example:

Theorem 2. *For every* $n \in \mathrm{N}_0 = \{0, 1, 2, 3, \ldots\}$,

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

i.e. $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.

Proof. **Base step** $n = 0$: LHS $= \sum_{k=1}^{0} k =$ the empty sum $=$ 0. RHS $= \frac{0 \cdot 1}{2} = 0$

So the claim is true for $n = 0$.

Induction step: Suppose

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \text{ for some } n \geq 0$$

We must show that

$$\sum_{k=1}^{n+1} k = \frac{(n+1)((n+1)+1)}{2}$$

$$
\begin{aligned}
\text{LHS} &= \sum_{k=1}^{n+1} k \\
&= \sum_{k=1}^{n} k + (n+1) \\
&= \frac{n(n+1)}{2} + (n+1) \text{ by the Induction hypothesis} \\
&= (n+1)\left(\frac{n}{2} + 1\right) \\
&= \frac{(n+1)(n+2)}{2} \\
\text{RHS} &= \frac{(n+1)((n+1)+1)}{2} \\
&= \frac{(n+1)(n+2)}{2} = \text{LHS}
\end{aligned}
$$

So by mathematical induction, $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ for all $n \in \mathbf{N}_0$.

$\square$

Proof by Contradiction

Assume the negation of what is claimed, and work toward a contradiction.

Theorem 3. *There is no rational number q such that $q^2 = 2$.*

Proof. Suppose $q^2 = 2$ where $q \in \mathbf{Q}$. Then we can write $q = \frac{m}{n}$ for some integers $m, n \in \mathbf{Z}$. Moreover, we can assume that m and n have no common factor; if they did, we could divide it out.

$$2 = q^2 = \frac{m^2}{n^2}$$

Therefore, $m^2 = 2n^2$, so m^2 is even.

We claim that m is even. If not, then m is odd, so $m = 2p + 1$ for some $p \in \mathbf{Z}$. Then

$$\begin{aligned} m^2 &= (2p+1)^2 \\ &= 4p^2 + 4p + 1 \\ &= 2(2p^2 + 2p) + 1 \end{aligned}$$

which is odd, contradiction. Therefore, m is even, so $m = 2r$ for some $r \in \mathbf{Z}$.

$$\begin{aligned} 4r^2 &= (2r)^2 \\ &= m^2 \\ &= 2n^2 \\ n^2 &= 2r^2 \end{aligned}$$

So n^2 is even, which implies (by the argument given above) that n is even. Therefore, $n = 2s$ for some $s \in \mathbf{Z}$, so m and n have a common factor, namely 2, contradiction. Therefore, there is no rational number q such that $q^2 = 2$. $\square$

Equivalence Relations

Definition 1. *A* binary relation *R from X to Y is a subset $R \subseteq X \times Y$. We write xRy if $(x, y) \in R$ and "not xRy" if $(x, y) \notin R$. $R \subseteq X \times X$ is a* binary relation on *X.*

Example: Suppose $f : X \to Y$ is a function from X to Y. The binary relation $R \subseteq X \times Y$ defined by

$$xRy \iff f(x) = y$$

is exactly the graph of the function f. A function can be considered a binary relation R from X to Y such that for each $x \in X$ there exists exactly one $y \in Y$ such that $(x, y) \in R$.

Example: Suppose $X = \{1, 2, 3\}$ and R is the binary relation on X given by $R = \{(1, 1), (2, 1), (2, 2), (3, 1), (3, 2), (3, 3)\}$. This is the binary relation "is weakly greater than," or $\geq$.

Equivalence Relations

Definition 2. *A binary relation* R *on* X *is*

(i) reflexive *if* $\forall x \in X, xRx$

(ii) symmetric *if* $\forall x, y \in X, xRy \Leftrightarrow yRx$

(iii) transitive *if* $\forall x, y, z \in X, (xRy \wedge yRz) \Rightarrow xRz$

Definition 3. *A binary relation* R *on* X *is an* equivalence relation *if it is reflexive, symmetric and transitive.*

Equivalence Relations

Definition 4. *Given an equivalence relation R on X, write*

$$[x] = \{y \in X : xRy\}$$

$[x]$ *is called the* equivalence class containing x.

The set of equivalence classes is the quotient *of X with respect to R, denoted X/R.*

Example: The binary relation $\geq$ on $\mathbf{R}$ is not an equivalence relation because it is not symmetric.

Example: Let $X = \{a, b, c, d\}$ and

$$R = \{(a, a), (a, b), (b, a), (b, b), (c, c), (c, d), (d, c), (d, d)\}$$

R is an equivalence relation (why?) and the equivalence classes of R are $\{a, b\}$ and $\{c, d\}$. $X/R = \{\{a, b\}, \{c, d\}\}$

Equivalence Relations

The equivalence classes of an equivalence relation form a *partition* of X: every element of X belongs to exactly one equivalence class.

Theorem 4. *Let R be an equivalence relation on X. Then $\forall x \in X, x \in [x]$. Given $x, y \in X$, either $[x] = [y]$ or $[x] \cap [y] = \emptyset$.*

Proof. If $x \in X$, then xRx because R is reflexive, so $x \in [x]$.

Suppose $x, y \in X$. If $[x] \cap [y] = \emptyset$, we're done. So suppose $[x] \cap [y] \neq \emptyset$. We must show that $[x] = [y]$, i.e. that the elements of $[x]$ are exactly the same as the elements of $[y]$.

Choose $z \in [x] \cap [y]$. Then $z \in [x]$, so xRz. By symmetry, zRx. Also $z \in [y]$, so yRz. Now choose $w \in [x]$. By definition, xRw. Since zRx and R is transitive, zRw. Since yRz, yRw by transitivity again. So $w \in [y]$, which shows that $[x] \subseteq [y]$.
Similarly, $[y] \subseteq [x]$, so $[x] = [y]$. $\square$

Cardinality

Definition 5. *Two sets* A, B *are* numerically equivalent *(or* have the same cardinality*) if there is a bijection* $f : A \to B$*, that is, a function* $f : A \to B$ *that is 1-1* $(a \neq a' \Rightarrow f(a) \neq f(a'))$*, and onto* $(\forall b \in B \; \exists a \in A$ *s.t.* $f(a) = b)$*.*

Example: $A = \{2, 4, 6, \ldots, 50\}$ is numerically equivalent to the set $\{1, 2, \ldots, 25\}$ under the function $f(n) = 2n$.

$B = \{1, 4, 9, 16, 25, 36, 49 \ldots\} = \{n^2 : n \in \mathbf{N}\}$ is numerically equivalent to $\mathbf{N}$.

Cardinality

A set is either finite or infinite. A set is *finite* if it is numerically equivalent to $\{1, \ldots, n\}$ for some n. A set that is not finite is *infinite*.

In particular, $A = \{2, 4, 6, \ldots, 50\}$ is finite, $B = \{1, 4, 9, 16, 25, 36, 49 \ldots\}$ is infinite.

A set is *countable* if it is numerically equivalent to the set of natural numbers $\mathbf{N} = \{1, 2, 3, \ldots\}$. An infinite set that is not countable is called *uncountable*.

Cardinality

Example: The set of integers $\mathbf{Z}$ is countable.

$$\mathbf{Z} = \{0, 1, -1, 2, -2, \ldots\}$$

Define $f : \mathbf{N} \to \mathbf{Z}$ by

$$\begin{aligned} f(1) &= 0 \\ f(2) &= 1 \\ f(3) &= -1 \\ &\vdots \\ f(n) &= (-1)^n \left\lfloor \frac{n}{2} \right\rfloor \end{aligned}$$

where $\lfloor x \rfloor$ is the greatest integer less than or equal to x. It is straightforward to verify that f is one-to-one and onto.

Cardinality

Theorem 5. *The set of rational numbers* $\mathbf{Q}$ *is countable.*

"Picture Proof":

$$\begin{aligned} \mathbf{Q} &= \left\{ \frac{m}{n} : m, n \in \mathbf{Z}, n \neq 0 \right\} \\ &= \left\{ \frac{m}{n} : m \in \mathbf{Z}, n \in \mathbf{N} \right\} \end{aligned}$$

		m								
		0		1		-1		2		-2
	1	0	$\rightarrow$	1		-1	$\rightarrow$	2		-2
			$\swarrow$		$\nearrow$		$\swarrow$		$\nearrow$	
	2	0		$\frac{1}{2}$		$-\frac{1}{2}$		1		-1
		$\downarrow$	$\nearrow$		$\swarrow$		$\nearrow$			
n	3	0		$\frac{1}{3}$		$-\frac{1}{3}$		$\frac{2}{3}$		$-\frac{2}{3}$
			$\swarrow$		$\nearrow$					
	4	0		$\frac{1}{4}$		$-\frac{1}{4}$		$\frac{1}{2}$		$-\frac{1}{2}$
		$\downarrow$	$\nearrow$							
	5	0		$\frac{1}{5}$		$-\frac{1}{5}$		$\frac{2}{5}$		$-\frac{2}{5}$

Go back and forth on upward-sloping diagonals, omitting the repeats:

$$
\begin{aligned}
f(1) &= 0 \\
f(2) &= 1 \\
f(3) &= \frac{1}{2} \\
f(4) &= -1 \\
&\vdots
\end{aligned}
$$

$f : \mathbf{N} \to \mathbf{Q}$, f is one-to-one and onto.