Problem 1.

Take any mapping f from a metric space X into a metric space Y. Prove that f is continuous if and only if $f(\overline{A}) \subseteq \overline{f(A)}$. (Hint: use the closed set characterization of continuity).

Solution

I make use of the following properties of images and pre-images of functions. For any sets $A \subset X, B \subset Y$:[1]

$$A \subset f^{-1}(f(A)), \;\; f(f^{-1}(B)) \subset B$$

Now for the proof:

($\implies$): Suppose f is continuous. Take any $A \subset X$. First note $A \subset f^{-1}(f(A)) \subset f^{-1}(\overline{f(A)})$. Then

$$\overline{A} \subset \overline{f^{-1}(\overline{f(A)})} = f^{-1}(\overline{f(A)})$$

because we know that $f^{-1}(\overline{f(A)})$ is closed from the continuity of f. Then take the image of both sides to get

$$f(\overline{A}) \subset f(f^{-1}(\overline{f(A)})) \subset \overline{f(A)}$$

where the final set inclusion follows from the properties above.

($\impliedby$): Suppose a function f satisfies $f(\overline{A}) \subseteq \overline{f(A)}$ for every set A. Take $C \subset Y$ closed. We want to show $D = f^{-1}(C)$ is closed. First note

$$\overline{f(D)} = \overline{f(f^{-1}(C))} \subset \overline{C} = C$$

Hence we have $f(\overline{D}) \subset C$. Then taking the pre-image of both sides gives

$$f^{-1}(f(\overline{D})) \subset f^{-1}(C) = D$$

From the properties above, $\overline{D} \subset f^{-1}(f(\overline{D}))$ and combining we reach $\overline{D} \subset D$. Thus D is closed, and f is continuous.

[1] The set inclusions may be proper; try to come up with examples.

Problem 2.

A function $f : X \to Y$ is *open* if for every open set $A \subset X$, its image $f(A)$ is also open. Show that any continuous open function from $\mathbb{R}$ into $\mathbb{R}$ (with the usual metric) is strictly monotonic.

Solution

Suppose the open mapping f is not strictly monotonic. So without loss of generality, for some $a < c < b \in \mathbb{R}$, we have $f(a) \leq f(c) \geq f(b)$.[2] Further, the compactness of $[a, b]$ and continuity of f gives us that $f([a, b])$ is compact. The extreme value theorem gives us that $M \equiv \sup f([a, b]) \in f([a, b])$. We have two cases:

- $f(a) = M$ or $f(b) = M$ in which case $f(c) = M$ and so $M \in f((a, b))$
- $f(a) < M$ and $f(b) < M$ so $\sup f([a, b]) = \sup f((a, b))$ and hence $M \in f((a, b))$.

In either case, $f((a, b))$ is not open, since the supremum of a set of real numbers cannot be an interior point of that set. This contradicts our assumption that f was an open mapping. Hence f is strictly monotonic.

[2]The other case is $f(a) \geq f(c) \leq f(b)$ and would use infimums rather than supremums.

Problem 3.

Suppose f, g are continuous functions from metric spaces (X, d) into (Y, ρ). Let E be a dense subset of X (in a metric space, a set A is dense in B if $\overline{A} \supset B$). Show that $f(E)$ is dense in $f(X)$. Further, if $f(x) = g(x)$ for every $x \in E$, then $f(x) = g(x)$ for every $x \in X$.

Solution

To show $f(E)$ is dense in $f(X)$, we need to show for every $y \in f(X)$, either $y \in f(E)$ or y is a limit point of E. So choose some $x \in X$ such that $f(x) = y$. Either $x \in E$ (in which case $f(x) = y \in f(E)$) or x is a limit point of E. In the latter case there exists a sequence $\{x_n\} \subset E$ such that $x_n \to x$. $x_n \in E \implies f(x_n) \in f(E)$ and continuity of f implies $f(x_n) \to f(x) = y$. Hence y is a limit point of $f(E)$.

Now suppose $f(x) = g(x)$ for every $x \in E$. Choose $x' \in X \setminus E$ (so x' is a limit point of E) and any sequence $\{x_n\} \subset E$ such that $x_n \to x'$. Then continuity guarantees that $f(x_n) \to f(x')$ and $g(x_n) \to g(x')$. But since $g(x_n) = f(x_n)$ for every $n \in \mathbb{N}$, the limit must be the same. So $f(x') = g(x')$.

Remark: This says that a continuous function is entirely determined by its values on any dense subset of its domain. So for example, a continuous real-valued function is determined by its values on the rationals.

Problem 4.

Show that in a metric space, a set is closed if and only if its intersection with any compact set is closed.

Solution

($\implies$): Let A be closed and C be any compact set. Let's show that $A \cap C$ contains its limit points. So suppose a is a limit point of $A \cap C$. We can find a sequence $\{a_n\} \subset A \cap C$ such that $a_n \to a$. Since A is closed we know $a \in A$. From the sequential characterization of compactness, we know that the sequence must contain some convergent subsequence $a_{n_k} \to a' \in C$. But any subsequence of a convergent sequence converges to the same limit, so $a' = a$ and hence $a \in A \cap C$.

($\impliedby$): For any set A, suppose $A \cap C$ is closed whenever C compact. Take any limit point a of A. Again find a sequence $\{a_n\} \subset A$ such that $a_n \to a$. We saw in section that the set $C = \cup_n \{a_n\} \cup \{a\}$, ie the set of all the sequence elements as well as its limit, is compact. So $A \cap C$ is closed, and since a is a limit point of $A \cap C$ we also have $a \in A \cap C \implies a \in A$.

Problem 5.

Show that a metric space X is connected if and only if every continuous function $f : X \to \{0, 1\}$ is constant.

Solution

It's easier to prove the equivalent statement: a metric space X is disconnected if and only if there exists a continuous function $f : X \to \{0, 1\}$ that is non-constant.

($\Longrightarrow$): Since X is disconnected, in section we saw that we can write $X = U \cup V$ where U, V are nonempty, open, and disjoint. Then we can define

$$f = \begin{cases} 0 & \text{if } x \in U \\ 1 & \text{if } x \in V \end{cases}$$

f is non-constant. Also we have $f^{-1}(\varnothing) = \varnothing$, $f^{-1}(\{0\}) = U$, $f^{-1}(\{1\}) = V$, and $f^{-1}(\{0, 1\}) = X$ are open, hence the pre-image of every open set is open which verifies f is continuous.

($\Longleftarrow$:) since f is non-constant, $f^{-1}(\{0\})$ and $f^{-1}(\{1\})$ are both nonempty; they are also both open by the continuity of f. Then since the codomain is defined as $\{0, 1\}$, we have $f^{-1}(\{0\}) \cap f^{-1}(\{1\}) = \varnothing$ and $f^{-1}(\{0\}) \cup f^{-1}(\{1\}) = X$. So we have two nonempty, disjoint open sets that partition X, so the space is disconnected.

Problem 6.

Let (X, d) be a compact metric space and let $\Phi(x) : X \to 2^X$ be a upper-hemicontinuous, compact-valued correspondence, such that $\Phi(x)$ is non-empty for every $x \in X$. Prove that there exists a compact non-empty subset K of X, such that $\Phi(K) \equiv \bigcup_{x \in K} \Phi(x) = K$.

Solution

There's a lot to show in this one. Let's start here:

Lemma. *Let (X, d) be a metric space and let $\Psi(x) : X \to 2^X$ be a upper-hemicontinuous, compact-valued and non-empty correspondence. If $K \subset X$ is compact, then $\Psi(K)$ is compact.*

Proof. We will use the sequential characterization of upper-hemicontinuity and compactness. Choose any sequence $\{y_n\} \subset \Psi(K)$. So for every y_n we can find some x_n such that $y_n \in \Psi(x_n)$. Compactness of K means we can find a convergent subsequence $x_{n_k} \to x_0 \in K$. Then consider the corresponding subsequence $\{y_{n_k}\}$. By the sequential characterization of compact-valued and upper-hemicontinuous correspondences we can find a convergent (sub)subsequence $y_{n_{k_j}} \to y_0 \in \Psi(x_0)$. But this (sub)subsequence is itself a subsequence of $\{y_n\}$, and $x_0 \in K \implies \Psi(x_0) \subset \Psi(K)$. Hence for an arbitrary sequence in $\Psi(K)$ we can find a convergent subsequence whose limit lies in $\Psi(K)$. Thus the set is sequentially compact, hence compact. □

Also, note that $A \subset B \implies \Psi(A) = \bigcup_{a \in A} \Psi(a) \subset \bigcup_{b \in B} \Psi(b) = \Psi(B)$ for any correspondence Ψ. So let's construct the following sequence of sets:

$$\begin{aligned} K_0 &= X \\ K_1 &= \Phi(K_0) \\ &\vdots \\ K_n &= \Phi(K_{n-1}) \\ &\vdots \end{aligned}$$

Using our Lemma, we can see inductively that that $K_0, K_1, \ldots$ are a sequence of nested, non-empty and compact sets. Then Cantor's intersection theorem tells us that $K = \bigcap_{n=0}^{\infty} K_n$ is non-empty. Since K is the intersection of closed sets, it is also closed. Then K is a closed subset of a compact metric space, so it is also compact.[3] Now I claim that $K = \Phi(K)$ otherwise why would I be doing all this?

First the easy direction: since $K \subset K_n$ for all n, we have $\Phi(K) \subset \Phi(K_n) = K_{n+1}$. Thus $\Phi(K) \subset K$. The other direction is more difficult, and the notation gets a bit cumbersome.

To show $K \subset \Phi(K)$, choose any $y_0 \in K$. Note for every n, we have $y_0 \in K_{n+1} = \Phi(K_n)$, so let's construct a sequence $\{x_n\}$ such that $x_n \in K_n$ and $y_0 \in \Phi(x_n)$. Since $\{x_n\} \subset K_0$, by compactness we can find a convergent subsequence $\{x_{n_j}\}$ with limit x_0. From how we have constructed the sequence, $\{x_n\}_{n \geq N}$ is entirely contained in K_N. But then for every N we

[3] In fact any closed subset of a compact *set* is compact.

can find some J such that $\{x_{n_j}\}_{j \geq J}$ is entirely contained in K_N. Hence x_0 is a limit point of every $K_N \implies x_0 \in K_N \; \forall N \implies x_0 \in K$.

Now finally, we have $y_0 \in \Phi(x_{n_j})$ for every n_j. Then this defines a constant sequence $y_{n_j} = y_0$, which of course converges to y_0 (along with all its subsequences). Using the sequential characterization of upper-hemicontinuous compact-valued correspondences, we know that $y_0 \in \Phi(x_0)$. Since we showed that $x_0 \in K$, we have $y_0 \in \Phi(K)$. y_0 was an arbitrary element of K, we have $K \subset \Phi(K)$.