Econ 204 (2012) - Final Solutions

08/13/2012

1. (20pts) Let $\Psi : \mathbb{R} \to 2^{\mathbb{R}}$ be the correspondence defined by

$$\Psi(x) = \begin{cases} \{0\} & \text{if } x \neq 0 \\ (-1, +\infty) & \text{if } x = 0 \end{cases}$$

   (a) Show that Ψ does not have a closed graph.

   Note that $(0, -1 + \frac{1}{n}) \in \text{graph } \Psi$ for all $n \in \mathbb{N}$ and $(0, -1 + \frac{1}{n}) \to (0, -1) \notin \text{graph } \Psi$. So Ψ does not have a closed graph.

   (b) Show that Ψ is upper hemicontinuous.

   Let $x \in \mathbb{R}$ and $V \subset \mathbb{R}$ such that $\Phi(x) \subset V$.

   Case 1: If $x \neq 0$, let $\delta = |x| > 0$ and $U = (x - \delta, x + \delta)$. Then, U is a neighborhood of x such that for all $x' \in U$, $x' \neq 0$, implying $\Psi(x') = \{0\} = \Psi(x) \subset V$. Therefore, Ψ is upper hemicontinuous at x.

   Case 2: If $x = 0$, then $U = \mathbb{R}$ is a neighborhood of 0 such that for all $x' \in U$, $\Psi(x') \subset (-1, +\infty) = \Psi(0) \subset V$. Therefore, Ψ is upper hemicontinuous at 0.

   (c) Show that Ψ is not lower hemicontinuous.

   Let $V = (0, +\infty)$. Note that $\Psi(0) \cap V = (0, +\infty) \neq \emptyset$, however for any neighborhood U of 0, there is $x' \in U$ such that $x' > 0$, i.e. $\Psi(x') \cap V = \{0\} \cap V = \emptyset$. Therefore, Ψ is not lower hemicontinuous at 0.

2. (25pts) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x) = x_1 \sin(x_2)$$

   (a) Find the critical points of f.

   Remember that $x \in \mathbb{R}^2$ is a critical point of f if and only if Rank $Df(x) < \min\{1, 2\} = 1$ if and only if

$$Df(x) = \left(\frac{\partial f}{\partial x_1}(x), \frac{\partial f}{\partial x_2}(x) \right) = (\sin(x_2), x_1 \cos(x_2)) = (0, 0)$$

   Therefore, the set of critical points are given by $\{(0, n\pi) : n \in \mathbb{Z}\}$. Note that $f(x) = 0$ at every critical point x of f.

(b) Give the second order Taylor expansion of f around each of its critical points.

$$D^2 f(x) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(x) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(x) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(x) & \frac{\partial^2 f}{\partial x_2^2}(x) \end{pmatrix} = \begin{pmatrix} 0 & \cos(x_2) \\ \cos(x_2) & -x_1 \sin(x_2) \end{pmatrix}$$

Fix a critical point $x = (0, n\pi)$. Then,

$$\begin{aligned} f(x+h) = f(h_1, n\pi + h_2) &= f(x) + Df(x)h + \frac{1}{2} h^T D^2 f(x) h + O(|h|^3) \\ &= \frac{1}{2} h^T D^2 f(x) h + O(|h|^3). \end{aligned}$$

If n is even, then

$$\begin{aligned} f(x+h) = f(h_1, n\pi + h_2) &= \frac{1}{2}(h_1, h_2) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + O(|h|^3) \\ &= h_1 h_2 + O(|h|^3). \end{aligned}$$

If n is odd, then

$$\begin{aligned} f(x+h) = f(h_1, n\pi + h_2) &= \frac{1}{2}(h_1, h_2) \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + O(|h|^3) \\ &= -h_1 h_2 + O(|h|^3). \end{aligned}$$

(c) Is any of the critical points a local maximizer or a local minimizer of f?

Take any critical point $x = (0, n\pi)$ of f. The characteristic polynomial of $D^2 f(x)$ is $\lambda^2 - 1 = 0$ (whether n is even or odd). Since $D^2 f(x)$ has one positive and one negative eigenvalue ($\lambda_1 = 1$ and $\lambda_2 = -1$), x is neither a local minimizer or a local maximizer of f.

3. (30pts) Let X denote the space of all bounded sequences of real numbers:

$$X = \{x = (x_1, x_2, \ldots) \in \mathbb{R}^{\mathbb{N}} : \sup\{|x_i| : i \in \mathbb{N}\} < +\infty\}$$

Note that X is a vector space over $\mathbb{R}$.[1]

[1] The scalar multiplication and vector addition operations on X are defined coordinatewise. That is, for every $\alpha \in \mathbb{R}$, and $x, y \in X$, the sequences $\alpha x \in X$ and $x + y \in X$ are defined by:

$$\forall n \in \mathbb{N} : \quad (\alpha x)_n = \alpha x_n \text{ and } (x+y)_n = x_n + y_n.$$

(a) For each $x \in X$, let $\|x\|_\infty = \sup\{|x_i| : i \in \mathbb{N}\}$. Show that $\|\cdot\|_\infty$ is a norm.

$\underline{\forall x \in X : \|x\|_\infty \in \mathbb{R}_+:}$

Since $x_i \geq 0$ for all $i \in \mathbb{N}$ and x is a bounded sequence $0 \leq \sup\{|x_i| : i \in \mathbb{N}\} < \infty$, so $\|x\|_\infty \in \mathbb{R}_+$.

$\underline{\forall x \in X : \|x\|_\infty = 0 \Leftrightarrow x = 0:}$

If x is the zero sequence then $\|x\| = \sup\{|x_i| : i \in \mathbb{N}\} = \sup\{0\} = 0$. If $\|x\|_\infty = 0$ then $|x_n| \leq \sup\{|x_i| : i \in \mathbb{N}\} = \|x\|_\infty = 0$ for all $n \in \mathbb{N}$, implying that x is the zero sequence.

$\underline{\forall x \in X, \alpha \in \mathbb{R} : \|\alpha x\|_\infty = |\alpha|\|x\|_\infty:}$

If $\alpha = 0$ the equality holds trivially. Suppose $\alpha \neq 0$. Since $\|x\|_\infty$ is an upper bound for the set $\{|x_i| : i \in \mathbb{N}\}$, $|\alpha|\|x\|_\infty$ is an upper bound for the set $\{|\alpha||x_i| : i \in \mathbb{N}\} = \{|\alpha x_i| : i \in \mathbb{N}\}$ implying:

$$\|\alpha x\|_\infty = \sup\{|\alpha x_i| : i \in \mathbb{N}\} \leq |\alpha|\|x\|_\infty \quad (*)$$

Applying Equation $(*)$ to $\frac{1}{\alpha}$ and αx, we also have $\|x\|_\infty = \|\frac{1}{\alpha}(\alpha x)\|_\infty \leq |\frac{1}{\alpha}|\|\alpha x\|_\infty$, also implying $|\alpha|\|x\|_\infty \leq \|\alpha x\|_\infty$.

$\underline{\forall x, y \in X : \|x + y\|_\infty \leq \|x\|_\infty + \|y\|_\infty:}$

For all $n \in \mathbb{N}$, $|x_n + y_n| \leq |x_n| + |y_n|$. Since $|x_n| \leq \|x\|_\infty$ and $|y_n| \leq \|x\|_\infty$, by definition of $\|\cdot\|_\infty$, we have that for all $n \in \mathbb{N}$

$$|x_n + y_n| \leq \|x\|_\infty + \|y\|_\infty.$$

That is, $\|x\|_\infty + \|y\|_\infty$ is an upper bound for the set $\{|x_n + y_n| : n \in \mathbb{N}\}$. Taking supremum over all $n \in \mathbb{N}$ in the l.h.s. of the above inequality gives the desired inequality.

(b) Let $T \in L(X, X)$ be defined by

$$(T(x))_n = x_n - x_{n+1} \text{ for every } x \in X \text{ and } n \in \mathbb{N}.$$

That is, the nth element of the sequence $T(x)$ is the difference $x_n - x_{n+1}$. Show that the linear map T is bounded and find its norm $\|T\|$.

Take any $x \in X$, and $n \in \mathbb{N}$:

$$|(T(x))_n| = |x_n - x_{n+1}| \leq |x_n| + |x_{n+1}| \leq 2\|x\|_\infty$$

Taking supremum over all $n \in N$ in the left hand side, we obtain

$$\|T(x)\|_\infty = \sup\{|(T(x))_n| : n \in \mathbb{N} \ \leq 2\|x\|_\infty$$

Therefore, T is bounded. Furthermore the above inequality implies that $\|T\| \leq 2$.

Now consider the sequence $\hat{x} = (1, -1, 0, 0, \ldots)$. Note that $T(\hat{x}) = (2, -1, 0, 0, \ldots)$. Then:

$$\|T\| \geq \frac{\|T(\hat{x})\|_\infty}{\|\hat{x}\|_\infty} = \frac{2}{1} = 2$$

So $\|T\| = 2$.

(c) Show that $\operatorname{Ker}(T) \cap \operatorname{Im}(T) = \{0\}$.

First note that $0 \in \operatorname{Ker}(T) \cap \operatorname{Im}(T)$ since $\operatorname{Ker}(T)$ and $\operatorname{Im}(T)$ are vector subspaces of X.

Note that $x \in \operatorname{Ker}(T)$ if and only if $0 = T(x) = (x_1 - x_2, x_2 - x_3, x_3 - x_4, \ldots)$ if and only if $x_1 = x_2 = x_3 = \ldots$. Hence $\operatorname{Ker}(T)$ consists of only constant sequences.

Now take any $y \in \operatorname{Ker}(T) \cap \operatorname{Im}(T)$. Since $y \in \operatorname{Ker}(T)$, there is $c \in \mathbb{R}$ such that $y_n = c$ for all $n \in \mathbb{N}$. Since $y \in \operatorname{Im}(T)$, there is $x \in X$ such that $T(x) = y$, i.e. $(T(x))_n = x_n - x_{n+1} = c$ for all $n \in N$. Note that:

$$(n-1)c = y_1 + y_2 + \ldots + y_{n-1} = (x_1 - x_2) + (x_2 - x_3) + \ldots + (x_{n-1} - x_n) = x_1 - x_n.$$

This implies that $(n-1)|c| \leq |x_1| + |x_n|$, i.e. $(n-1)|c| - |x_1| \leq |x_n|$. By the definition of supremum:

$$\forall n \in \mathbb{N} : (n-1)|c| - |x_1| \leq |x_n| \leq \sup\{|x_i| : i \in \mathbb{N}\}.$$

Therefore, the supremum above is finite only if $c = 0$. This proves that $y = 0$. So $\operatorname{Ker}(T) \cap \operatorname{Im}(T) = \{0\}$.

4. (25pts) Let (X, d) be a metric space. Let $(A_i)_{i \in \mathbb{N}}$ be a sequence of nonempty compact subsets of X such that $A_i \supset A_{i+1}$ for all $i \in \mathbb{N}$.

(a) Prove that if $\cap_{i=1}^\infty A_i = \emptyset$, then $A_1 \subset \cup_{i=1}^\infty (X \setminus A_i)$.

Suppose $\cap_{i=1}^\infty A_i = \emptyset$. By de Morgan's law:

$$\cup_{i=1}^\infty (X \setminus A_i) = X \setminus (\cap_{i=1}^\infty A_i) = X \setminus \emptyset = X \supset A_1.$$

(b) Use your finding in part (a) to prove that $\cap_{i=1}^\infty A_i \neq \emptyset$.

For all $i \in \mathbb{N}$, since A_i is compact, it is closed, implying that $X \setminus A_i$ is open. Suppose $\cap_{i=1}^{\infty} A_i = \emptyset$. From above $A_1 \subset \cup_{i=1}^{\infty}(X \setminus A_i)$, i.e. $\{X \setminus A_i\}_{i \in \mathbb{N}}$ is an open covering of A_1. Since A_1 is compact there is a finite subcovering $A_{i_1}, \ldots, A_{i_k}$. Suppose wlog $i_1 < i_2 < \ldots < i_k$. Then,

$$A_1 \subset \cup_{l=1}^{k}(X \setminus A_{i_l}) = X \setminus \left(\cap_{l=1}^{k} A_{i_l}\right) = X \setminus A_{i_k}.$$

by de Morgan's law and since $A_{i_1} \supset A_{i_2} \supset \ldots \supset A_{i_k}$. The above inclusion implies that $A_{i_k} \subset A_1 \subset X \setminus A_{i_k}$, which contradicts nonemptiness of A_{i_k}.

5. (Bonus, extra 20pts) Prove that every convex subset of $\mathbb{R}^n$ is connected.

Let $Y \subset \mathbb{R}^n$ be convex. Suppose that Y is not connected, i.e., there exist $A, B \subset \mathbb{R}^n$ such that A and B are nonempty, $Y = A \cup B$, and $\bar{A} \cap B = A \cap \bar{B} = \emptyset$. Let $x \in A$ and $y \in B$, and define the function $f : [0,1] \to \mathbb{R}^n$ by $f(\alpha) = \alpha x + (1-\alpha)y$. Note that f is continuous since if $\alpha_n \to \alpha$ in $[0,1]$, then $f(\alpha_n) = \alpha_n x + (1-\alpha_n)y \to \alpha x + (1-\alpha)y = f(\alpha)$. Define $A' = f^{-1}(A)$ and $B' = f^{-1}(B)$.

Note that $1 \in A'$ since $f(1) = x \in A$ and $0 \in B'$ since $f(0) = y \in B$. Therefore, A' and B' are nonempty.

For any $\alpha \in [0,1]$, since Y is convex, $f(\alpha) = \alpha x + (1-\alpha)y \in Y$, so $\alpha \in f^{-1}(Y) = f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B) = A' \cup B'$. Therefore, $[0,1] = A' \cup B'$.

Since f is continuous and $\bar{A}$ is closed in $\mathbb{R}^n$, $f^{-1}(\bar{A})$ is closed in $[0,1]$. Furthermore, $A' \subset f^{-1}(\bar{A})$, so $\bar{A}' \subset f^{-1}(\bar{A})$. Then,

$$\bar{A}' \cap B' \subset f^{-1}(\bar{A}) \cap f^{-1}(B) = f^{-1}(\bar{A} \cap B) = f^{-1}(\emptyset) = \emptyset.$$

The proof that $A' \cap \bar{B}' = \emptyset$ is similar. This implies that $[0,1]$ is not connected, a contradiction. Therefore, Y is connected.