Econ 204 (2012) - Final

08/13/2012

Instructions: This is a closed book exam. You have 3 hours. The weight of each question is indicated next to it. Write clearly, explain your answers, and be concise. You may use any result from class. Good luck!

1. (20pts) Let $\Psi : \mathbb{R} \to 2^{\mathbb{R}}$ be the correspondence defined by

$$\Psi(x) = \begin{cases} \{0\} & \text{if } x \neq 0 \\ (-1, +\infty) & \text{if } x = 0 \end{cases}$$

   (a) Show that Ψ does not have a closed graph.

   (b) Show that Ψ is upper hemicontinuous.

   (c) Show that Ψ is not lower hemicontinuous.

2. (25pts) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x) = x_1 \sin(x_2)$$

   (a) Find the critical points of f.

   (b) Give the second order Taylor expansion of f around each of its critical points.

   (c) Is any of the critical points a local maximizer or a local minimizer of f?

3. (30pts) Let X denote the space of all bounded sequences of real numbers:

$$X = \{x = (x_1, x_2, \ldots) \in \mathbb{R}^{\mathbb{N}} : \sup\{|x_i| : i \in \mathbb{N}\} < +\infty\}$$

   Note that X is a vector space over $\mathbb{R}$.[1]

   (a) For each $x \in X$, let $\|x\|_\infty = \sup\{|x_i| : i \in \mathbb{N}\}$. Show that $\|\cdot\|_\infty$ is a norm.

[1] The scalar multiplication and vector addition operations on X are defined coordinatewise. That is, for every $\alpha \in \mathbb{R}$, and $x, y \in X$, the sequences $\alpha x \in X$ and $x + y \in X$ are defined by:

$$\forall n \in \mathbb{N} : \quad (\alpha x)_n = \alpha x_n \text{ and } (x+y)_n = x_n + y_n.$$

(b) Let $T \in L(X, X)$ be defined by

$$(T(x))_n = x_n - x_{n+1} \text{ for every } x \in X \text{ and } n \in \mathbb{N}.$$

That is, the nth element of the sequence $T(x)$ is the difference $x_n - x_{n+1}$. Show that the linear map T is bounded and find its norm $\|T\|$.

(c) Show that $\mathrm{Ker}(T) \cap \mathrm{Im}(T) = \{0\}$.

4. (25pts) Let (X, d) be a metric space. Let $(A_i)_{i \in \mathbb{N}}$ be a sequence of nonempty compact subsets of X such that $A_i \supset A_{i+1}$ for all $i \in \mathbb{N}$.

   (a) Prove that if $\cap_{i=1}^{\infty} A_i = \emptyset$, then $A_1 \subset \cup_{i=1}^{\infty}(X \setminus A_i)$.

   (b) Use your finding in part (a) to prove that $\cap_{i=1}^{\infty} A_i \neq \emptyset$.

5. (Bonus, extra 20pts) Prove that every convex subset of $\mathbb{R}^n$ is connected.