Econ 204 (2013) - Final Solutions

08/21/2013

1. (15pts) Define $F : \mathbb{R}^2 \to \mathbb{R}$ by:

$$F(x, a) = x^3 - xa + \frac{1}{3}a^2 - 1 \qquad x, a \in \mathbb{R}.$$

For each of the following (x_0, a_0) values, state whether you can use the Implicit Function Theorem to conclude that there exist open sets $U, W \subset \mathbb{R}$ such that $x_0 \in U$, $a_0 \in W$, and a C^1 function $g : W \to U$ satisfying:

$$\forall a \in W : \quad F(g(a), a) = 0.$$

If your answer is yes, find $g'(a_0)$.

(a) $(x_0, a_0) = (1, 1)$.

(b) $(x_0, a_0) = (1, 3)$.

(c) $(x_0, a_0) = (1, 0)$.

Solution: The function F is a polynomial, it is therefore C^1. To see whether the Implicit Function Theorem can be applied to reach the desired conclusion, we only have to check the remaining two conditions of the Theorem: (i) (x_0, a_0) solves the equation, i.e. $F(x_0, a_0) = 0$, and (ii) x_0 is a regular point of $F(\cdot, a_0)$, i.e. $D_x F(x_0, a_0) = 3x_0^2 - a_0 \neq 0$.

(a) $F(1, 1) = -\frac{2}{3} \neq 0$, so $(1, 1)$ doesn't solve our equation, hence the Implicit Function Theorem can not be applied.

(b) $F(1, 3) = 0$, but $D_x F(1, 3) = 3 - 3 = 0$. So, (1,3) solves the equation but 1 is not a regular point of $F(\cdot, 3)$, hence the Implicit Function Theorem can not be applied.

(c) $F(1, 0) = 0$, and $D_x F(1, 0) = 3 \neq 0$. In this case, all the conditions of the Implicit Function Theorem hold, so we can apply it to obtain the desired conclusion. Furthermore, note that $D_a F(x, a) = -x + \frac{2}{3}a$, implying $D_a F(1, 0) = -1$, so

$$g'(0) = -[D_x F(1, 0)]^{-1}[D_a F(1, 0)] = -\frac{1}{3}(-1) = \frac{1}{3}.$$

2. (20pts) Find the solution $y : \mathbb{R} \to \mathbb{R}^2$ of the following initial value problem:

$$\begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} \text{ and } \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

You can leave the solution in the form of a product of matrices and a matrix inverse.[1] Illustrate the qualitative properties of $y(t)$ on the phase plane diagram.

Solution: Consider the matrix

$$M = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}.$$

The characteristic polynomial of M is given by $(1-\lambda)(1-\lambda) - 4 = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1)$. Therefore, the eigenvalues of M are $\lambda_1 = 3$ and $\lambda_2 = -1$. Since the eigenvalues are distinct, by Theorem 8 from lecture 9, the eigenvalues have linearly independent eigenvectors which constitute a basis for $\mathbb{R}^2$. Solving for v_i in $Mv_i = \lambda_i v_i$ for $i = 1, 2$, two such eigenvectors are:

$$v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ and } v_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

Let U denote the standard basis of $\mathbb{R}^2$ and let $V = \{v_1, v_2\}$ be the new basis consisting of eigenvectors of M. Define the matrices

$$A := \begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix} \text{ and } \Lambda := \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}.$$

That is, $A = [v_1, v_2]$ is the matrix whose columns are the eigenvectors, and Λ is the diagonal matrix of eigenvalues. Then, A changes basis from V to U; A^{-1} changes basis from U to V; and M can be diagonalized as $M = A\Lambda A^{-1}$.

By Theorem 2 from Lecture 14, the solution to this linear initial value problem is given by $y(t) = Ae^{t\Lambda}A^{-1}y(0)$:

$$y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}.$$

The stationary point of this linear differential equation is $0 \in \mathbb{R}^2$. In the phase plane diagram, you should draw the V-coordinates where the origin is 0, and

[1] That is, you do not have to carry out the matrix multiplication and matrix inversion.

the axes are in the directions of the eigenvectors. For any point in $\mathbb{R}^2$, if the v_1 coordinate is nonzero, then it will diverge (preserving its sign) since the real part of the corresponding eigenvalue λ_1 is strictly positive. The v_2 coordinate will converge to zero (again preserving its sign) since the real part of the corresponding eigenvalue λ_2 is strictly negative.

3. (15pts) Let X be a finite dimensional vector space over a field $\mathbb{F}$. Let W be a vector subspace of X. Remember the definition of the set:

$$[x] := \{y \in X : x - y \in W\} \qquad \text{for all } x \in X.$$

Consider the function $T : X \to X/W$ defined by $T(x) = [x]$ for any $x \in X$. Show that T is linear.[2] Use the Rank-Nullity Theorem to conclude:

$$\dim(X) = \dim(W) + \dim(X/W).$$

Solution: Take any $\alpha, \beta \in \mathbb{F}$ and $x, y \in X$. Then,

$$T(\alpha x + \beta y) = [\alpha x + \beta y] = [\alpha x] + [\beta y] = \alpha[x] + \beta[y] = \alpha T(x) + \beta T(y)$$

where the first and last equalities follow from the definition of T; and the second and third equalities follow from the definitions of the operations $+, \cdot$ in X/W respectively. Therefore, T is linear.

Note that $[0]$ is the vector additive identity in X/W. Note also that for all $x \in X$, $T(x) = [0]$ iff $[x] = [0]$ iff $x = x - 0 \in W$. Therefore,

$$\mathrm{Ker}(T) = \{x \in X : T(x) = [0]\} = W$$

Furthermore $\mathrm{Im}(T) = X/W$, because for any $[x] \in X/W$, $T(x) = [x]$. The Rank-Nullity Theorem states that when X is finite dimensional:

$$\dim(X) = \dim(\mathrm{Ker}(T)) + \dim(\mathrm{Im}(T)).$$

In this case, $\mathrm{Ker}(T) = W$ and $\mathrm{Im}(T) = X/W$ implying the desired conclusion.

4. Consider the L-shaped set $Y = (\{0\} \times [0,1]) \cup ([0,1] \times \{0\})$ in $\mathbb{R}^2$. Suppose that $f : Y \to Y$ is a continuous function.

[2] You can assume without proof that the vector space operations $+, \cdot$ in X/W given by $[x+y] := [x] + [y]$ and $[\alpha x] := \alpha[x]$ for all $x, y \in X$ and $\alpha \in \mathbb{F}$ are well-defined. That is, for all $x, y, x', y' \in X$ and $\alpha, \alpha' \in \mathbb{F}$: (i) $[x+y] = [x'+y'] \Rightarrow [x] + [y] = [x'] + [y']$ and (ii) $[\alpha x] = [\alpha' x'] \Rightarrow \alpha[x] = \alpha'[x']$.

(a) (5pts) Can you directly use Brouwer's Fixed Point Theorem to conclude that f has a fixed point? Explain your answer in at most two sentences.

Solution: Brouwer's Fixed Point Theorem can not be applied to f directly to conclude that f has a fixed point, because Y is not a convex subset of $\mathbb{R}^2$: $(0,1),(1,0) \in Y$ but $(\frac{1}{2},\frac{1}{2}) = \frac{1}{2}(0,1)+\frac{1}{2}(1,0) \notin Y$.

(b) (10pts) Remember that a homeomorphism between two metric spaces is a continuous bijection with a continuous inverse. Let $X := [-1,1]$ be the closed interval of the real line. Specify a homeomorphism $g : X \to Y$.[3] Use g and Brouwer's Fixed Point Theorem to show that f has a fixed point.

Solution: One such homeomorphism g is defined by:

$$g(x) = \begin{cases} (x,0) & \text{if } x \in [0,1] \\ (0,-x) & \text{if } x \in [-1,0) \end{cases}$$

Consider the function $g^{-1} \circ f \circ g : X \to X$. Note that $g^{-1} \circ f \circ g$ is continuous as the composition of continuous functions, and X is a compact convex subset of $\mathbb{R}$, therefore by the Brouwer's Fixed Point Theorem, $g^{-1} \circ f \circ g$ has a fixed point x^*. That is, $x^* = (g^{-1} \circ f \circ g)(x^*)$. If we apply g to both sides of this equality we obtain:

$$g(x^*) = (g \circ g^{-1} \circ f \circ g)(x^*) = (f \circ g)(x^*) = f(g(x^*)).$$

Therefore, $y^* := g(x^*)$ is a fixed point of f.

5. (20pts) Let (Y,ρ) be a metric space. Let $A \subset Y$ be a compact set and $\{y_n\}$ be a sequence in Y. Assume that for every open set V with $A \subset V$, there is $N \in \mathbb{N}$ such that $y_n \in V$ for all $n > N$. Show that $\{y_n\}$ has a subsequence that converges to a point in A.

Solution: For each $k \in \mathbb{N}$, let $V_k = \bigcup_{x \in A} B_{\frac{1}{k}}(x)$. Note that V_k is open and $A \subset V_k$. We will first inductively define a subsequence $\{y_{n_k}\}$ such that $y_{n_k} \in V_k$ for all $k \in \mathbb{N}$:

Step 1: By assumption, there is $N \in \mathbb{N}$ such that $y_n \in V_1$ for all $n > N$. Fix any $n_1 > N$. Note that $y_{n_1} \in V_1$.

[3] Make sure that the function g you specify is such that: (i) g is a bijection, (ii) g is continuous, and (iii) g^{-1} is continuous; however, you do not have to supply the proofs of (i)–(iii).

Step $k > 1$: Assume that n_{k-1} is already defined. By assumption, there is $N \in \mathbb{N}$ such that $y_n \in V_k$ for all $n > N$. Fix any $n_k > \max\{N, n_{k-1}\}$. Note that $n_k > n_{k-1}$ and $y_{n_k} \in V_k$.

Since $y_{n_k} \in V_k$, there is $x_k \in A$ such that $y_{n_k} \in B_{\frac{1}{k}}(x_k)$, i.e. $\rho(y_{n_k}, x_k) < \frac{1}{k}$. Since A is compact and $\{x_k\} \subset A$, there is a subsequence $\{x_{k_r}\}$ such that $x_{k_r} \to x^*$ for some $x^* \in A$.

Let $\epsilon > 0$ be given. Since $x_{k_r} \to x^*$, there exists $R_1 \in \mathbb{N}$ such that for all $r > R_1$ we have $\rho(x_{k_r}, x^*) < \epsilon/2$. Also there exists $R_2 \in \mathbb{N}$ such that $1/R_2 < \epsilon/2$. Set $R = \max\{R_1, R_2\}$. Then for all $r > R$:

$$\rho(y_{n_{k_r}}, x^*) \leq \rho(y_{n_{k_r}}, x_{k_r}) + \rho(x_{k_r}, x^*) < 1/k_r + \epsilon/2 = \epsilon/2 + \epsilon/2 = \epsilon$$

since $k_r \geq r > R \geq R_2 \geq 2/\epsilon$. Since $\epsilon > 0$ was arbitrary, the subsequence $y_{n_{k_r}}$ also converges to x^*.

6. (15pts) Prove the following sequential characterization of upper hemi continuity for compact-valued correspondences. You can assume without proof the statement you are asked to show in Question 5.

Theorem 1 *Suppose (X, d) and (Y, ρ) are metric spaces. A compact-valued correspondence $\Psi : X \to 2^Y$ is uhc at $x_0 \in X$ if and only if, for every sequence $\{x_n\} \subset X$ with $x_n \to x_0$, and every sequence $\{y_n\} \subset Y$ such that $y_n \in \Psi(x_n)$ for every $n \in \mathbb{N}$, there is a convergent subsequence $\{y_{n_k}\}$ such that $\lim y_{n_k} \in \Psi(x_0)$.*

Solution: "$\Leftarrow$": We will prove this direction by contraposition. Suppose that Ψ is not uhc at x_0. Then, there is an open set $V \subset Y$ with $\Psi(x_0) \subset V$, such that for any open set $U \subset X$ with $x_0 \in U$, there is $x \in U$ for which $\Psi(x)$ is not a subset of V. In particular, for each $n \in \mathbb{N}$, we can choose a point $x_n \in B_{\frac{1}{n}}(x_0)$ such that $\Psi(x_n)$ is not a subset of V, and a point $y_n \in \Psi(x_n) \setminus V$. Take any $y \in \Psi(x_0)$. Note that there is no subsequence of $\{y_n\}$ converging to y, because V is an open set such that $y \in \Psi(x_0) \subset V$ and $y_n \notin V$ for all $n \in \mathbb{N}$. As a result, the sequences $\{x_n\} \subset X$ and $\{y_n\} \subset Y$ are such that $x_n \to x_0$, and $y_n \in \Psi(x_n)$ for every $n \in \mathbb{N}$, but there is no convergent subsequence $\{y_{n_k}\}$ whose limit is in $\Psi(x_0)$.

"$\Rightarrow$": Suppose that Ψ is uhc at x_0, $\{x_n\} \subset X$ with $x_n \to x_0$, and $\{y_n\}$ is such that $y_n \in \Psi(x_n)$ for every $n \in \mathbb{N}$. Let $A := \Psi(x_0)$. A is compact since Ψ is compact-valued. Take any open set $V \subset Y$ such that $A = \Psi(x_0) \subset V$. By uhc of Ψ at x_0, there is an open set $U \subset X$ such that $x_0 \in U$ and for all $x \in U$, $\Psi(x) \subset V$. Since $x_n \to x_0$, there is $N \in \mathbb{N}$ such that for all $n > N$, $x_n \in U$, i.e. $y_n \in \Psi(x_n) \subset V$. Therefore, A is a compact set and for any open set V with $A \subset V$, there is $N \in \mathbb{N}$ such that $y_n \in V$ for all $n > N$. By Question 5, this implies that $\{y_n\}$ has a subsequence that converges to a point in $A = \Psi(x_0)$.

7. (Bonus, extra 20pts) Let (X, d) be a metric space and let $A \subset X$ be a totally bounded set. Show that every sequence in A has a Cauchy subsequence.

   Digression: In our solution to this question, we will construct a subsequence of a subsequence of a subsequence, ad infinitum; which might become notationally cumbersome because of having to use a subscript of a subscript of a subscript, ad infinitum. Therefore, it will be convenient to represent subsequences in the following alternative way. Fix a sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$. We will associate every subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$, with the infinite set $N \subset \mathbb{N}$ defined by $N = \{n_k : k \in \mathbb{N}\}$. It can be verified that this mapping, from the subsequences of $\{x_n\}_{n\in\mathbb{N}}$ to the collection of infinite subsets of $\mathbb{N}$, is a bijection.[4]

   Solution:

   Part 1: Suppose that $\{x_n\}_{n\in\mathbb{N}} \subset A$. In Part 2 of the solutions, we will construct a sequence of subsequences $\{N_k\}_{k\in\mathbb{N}}$ (i.e. for each $k \in \mathbb{N}$, N_k is an infinite subset of $\mathbb{N}$) satisfying the following two conditions:

   (a) $N_k \supset N_{k+1}$ (i.e. N_{k+1} is a subsequence of N_k) for all $k \in \mathbb{N}$.

   (b) For all $k \in \mathbb{N}$ and $n, m \in N_k$: $d(x_n, x_m) < \frac{1}{k}$.

   Given $\{N_k\}_{k\in\mathbb{N}}$ satisfying (a) and (b), inductively define a new subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ such that $n_k \in N_k$ for all $k \in \mathbb{N}$, by:

   Step 1*:* Let $n_1 := \min N_1$. Note that $n_1 \in N_1$.

   Step $k > 1$*:* Assume that n_{k-1} is already defined. Let $n_k := \min N_k \backslash \{1, 2, 3, \ldots, n_{k-1}\}$. Note that $n_k > n_{k-1}$ and $n_k \in N_k$.

   See Figure 1 for an illustration of the construction of the subsequence $N = \{n_k : k \in \mathbb{N}\}$.

[4] In particular, the inverse mapping, from the collection of infinite subsets of $\mathbb{N}$ to the subsequences of $\{x_n\}_{n\in\mathbb{N}}$, associates every infinite $N \subset \mathbb{N}$, with the subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ where n_k is defined as the kth smallest element of the set N.

$\mathbb{N}$	1	2	3	4	5	6	7	8	9	$\cdots$
N_1	**1**	2	3	4		6	7		9	$\cdots$
N_2		**2**	3	4		6			9	$\cdots$
N_3		2	**3**						9	$\cdots$
N_4			3			**6**			9	$\cdots$
N_5			3						**9**	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Figure 1. Construction of the subsequence $N = \{1, 2, 3, 6, 9, \ldots\}$

To see that the subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ defined above is Cauchy, take any $\epsilon > 0$. Let $k \in \mathbb{N}$ such that $\frac{1}{k} < \epsilon$. Then, for any $i, j > k$, $n_i \in N_i \subset N_k$ and $n_j \in N_j \subset N_k$. Therefore by condition (b), for any $i, j > k$

$$d(x_{n_i}, x_{n_j}) < \frac{1}{k} < \epsilon.$$

Since $\epsilon > 0$ was arbitrary, the subsequence $\{x_{n_k}\}_{n\in\mathbb{N}}$ is Cauchy.

Part 2: *Inductive definition of the sequence of subsequences* $\{N_k\}_{k\in\mathbb{N}}$ *satisfying conditions* (a) *and* (b):

Step 1*:* Since A is totally bounded there is a finite subset $A_1 \subset A$ such that $A \subset \bigcup_{x\in A_1} B_{\frac{1}{2}}(x)$. Since $\{x_n\}_{n\in\mathbb{N}} \subset A \subset \bigcup_{x\in A_1} B_{\frac{1}{2}}(x)$,

$$\mathbb{N} = \{n \in \mathbb{N} : x_n \in \bigcup_{x\in A_1} B_{\frac{1}{2}}(x)\} = \bigcup_{x\in A_1} \{n \in \mathbb{N} : x_n \in B_{\frac{1}{2}}(x)\}$$

Since the last union is the union of finitely many sets, one of those sets, say $\{n \in \mathbb{N} : x_n \in B_{\frac{1}{2}}(x)\}$ is infinite. Define the subsequence $N_1 := \{n \in \mathbb{N} : x_n \in B_{\frac{1}{2}}(x)\}$. Note that for any $n, m \in N_1$:

$$d(x_n, x_m) \le d(x_n, x) + d(x, x_m) < \frac{1}{2} + \frac{1}{2} = 1.$$

Step $k > 1$*:* Assume that the subsequence N_{k-1} is already defined. Since A is totally bounded there is a finite subset $A_k \subset A$ such that $A \subset \bigcup_{x\in A_k} B_{\frac{1}{2k}}(x)$. Since $\{x_n\}_{n\in N_{k-1}} \subset A \subset \bigcup_{x\in A_k} B_{\frac{1}{2k}}(x)$,

$$N_{k-1} = \{n \in N_{k-1} : x_n \in \bigcup_{x\in A_k} B_{\frac{1}{2k}}(x)\} = \bigcup_{x\in A_k} \{n \in N_{k-1} : x_n \in B_{\frac{1}{2k}}(x)\}$$

Since the last union is the union of finitely many sets, one of those sets, say $\{n \in N_{k-1} : x_n \in B_{\frac{1}{2k}}(x)\}$ is infinite. Define the subsequence $N_k := \{n \in N_{k-1} : x_n \in B_{\frac{1}{2k}}(x)\}$. Note that $N_k \subset N_{k-1}$ and for any $n, m \in N_k$:

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \frac{1}{2k} + \frac{1}{2k} = \frac{1}{k}.$$