Econ 204 (2013) - Final

08/21/2013

Instructions: This is a closed book exam. You have 3 hours. The weight of each question is indicated next to it. Write clearly, explain your answers, and be concise. You may use any result from class unless you are explicitly asked to prove it. Good luck!

1. (15pts) Define $F : \mathbb{R}^2 \to \mathbb{R}$ by:

$$F(x, a) = x^3 - xa + \frac{1}{3}a^2 - 1 \qquad x, a \in \mathbb{R}.$$

For each of the following (x_0, a_0) values, state whether you can use the Implicit Function Theorem to conclude that there exist open sets $U, W \subset \mathbb{R}$ such that $x_0 \in U$, $a_0 \in W$, and a C^1 function $g : W \to U$ satisfying:

$$\forall a \in W : \quad F(g(a), a) = 0.$$

If your answer is yes, find $g'(a_0)$.

   (a) $(x_0, a_0) = (1, 1)$.

   (b) $(x_0, a_0) = (1, 3)$.

   (c) $(x_0, a_0) = (1, 0)$.

2. (20pts) Find the solution $y : \mathbb{R} \to \mathbb{R}^2$ of the following initial value problem:

$$\begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} \text{ and } \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

You can leave the solution in the form of a product of matrices and a matrix inverse.[1] Illustrate the qualitative properties of $y(t)$ on the phase plane diagram.

3. (15pts) Let X be a finite dimensional vector space over a field $\mathbb{F}$. Let W be a vector subspace of X. Remember the definition of the set:

$$[x] := \{y \in X : x - y \in W\} \qquad \text{for all } x \in X.$$

[1] That is, you do not have to carry out the matrix multiplication and matrix inversion.

Consider the function $T : X \to X/W$ defined by $T(x) = [x]$ for any $x \in X$. Show that T is linear.[2] Use the Rank-Nullity Theorem to conclude:

$$\dim(X) = \dim(W) + \dim(X/W).$$

4. Consider the L-shaped set $Y = (\{0\} \times [0,1]) \cup ([0,1] \times \{0\})$ in $\mathbb{R}^2$. Suppose that $f : Y \to Y$ is a continuous function.

   (a) (5pts) Can you directly use Brouwer's Fixed Point Theorem to conclude that f has a fixed point? Explain your answer in at most two sentences.

   (b) (10pts) Remember that a homeomorphism between two metric spaces is a continuous bijection with a continuous inverse. Let $X := [-1, 1]$ be the closed interval of the real line. Specify a homeomorphism $g : X \to Y$.[3] Use g and Brouwer's Fixed Point Theorem to show that f has a fixed point.

5. (20pts) Let (Y, ρ) be a metric space. Let $A \subset Y$ be a compact set and $\{y_n\}$ be a sequence in Y. Assume that for every open set V with $A \subset V$, there is $N \in \mathbb{N}$ such that $y_n \in V$ for all $n > N$. Show that $\{y_n\}$ has a subsequence that converges to a point in A.

6. (15pts) Prove the following sequential characterization of upper hemi continuity for compact-valued correspondences. You can assume without proof the statement you are asked to show in Question 5.

   Theorem 1 *Suppose (X, d) and (Y, ρ) are metric spaces. A compact-valued correspondence $\Psi : X \to 2^Y$ is uhc at $x_0 \in X$ if and only if, for every sequence $\{x_n\} \subset X$ with $x_n \to x_0$, and every sequence $\{y_n\} \subset Y$ such that $y_n \in \Psi(x_n)$ for every $n \in \mathbb{N}$, there is a convergent subsequence $\{y_{n_k}\}$ such that $\lim y_{n_k} \in \Psi(x_0)$.*

7. (Bonus, extra 20pts) Let (X, d) be a metric space and let $A \subset X$ be a totally bounded set. Show that every sequence in A has a Cauchy subsequence.

[2] You can assume without proof that the vector space operations $+, \cdot$ in X/W given by $[x + y] := [x] + [y]$ and $[\alpha x] := \alpha[x]$ for all $x, y \in X$ and $\alpha \in \mathbb{F}$ are well-defined. That is, for all $x, y, x', y' \in X$ and $\alpha, \alpha' \in \mathbb{F}$: (i) $[x + y] = [x' + y'] \Rightarrow [x] + [y] = [x'] + [y']$ and (ii) $[\alpha x] = [\alpha' x'] \Rightarrow \alpha[x] = \alpha'[x']$.

[3] Make sure that the function g you specify is such that: (i) g is a bijection, (ii) g is continuous, and (iii) g^{-1} is continuous; however, you do not have to supply the proofs of (i)–(iii).