Econ 204 (2014) - Final Solutions
08/20/2014

1. (10pts) Suppose that A is a closed set in a metric space, and $x \notin A$. Show that

$$d(x, A) \equiv \inf\{d(x, y) : y \in A\} > 0.$$

Solution: Since A is closed, $X \setminus A$ is open. Therefore, $x \in X \setminus A$ impies that there exists $\epsilon > 0$ such that $B_\epsilon(x) \subset X \setminus A$. So for any $y \in A$, $y \notin B_\epsilon(x)$, i.e., $d(x, y) \geq \epsilon$. This proves that ϵ is a lower bound for the set $\{d(x, y) : y \in A\}$. Therefore, $\inf\{d(x, y) : y \in A\} \geq \epsilon > 0$.

2. (15pts) Let X and Y be (not necessarily finite dimensional) vector spaces over a field $\mathbb{F}$, and let $T \in L(X, Y)$ be such that $\operatorname{Ker} T = \{0\}$. Show that if V is a Hamel basis of X, then $T(V) \equiv \{T(v) : v \in V\}$ is a Hamel basis of $\operatorname{Im} T$.

Solution: Let's first show that $T(V)$ spans $\operatorname{Im} T$. Take any $y \in \operatorname{Im} T$, let $x \in X$ be such that $T(x) = y$. Since V is a Hamel basis for X, it spans X, so there exists finitely many basis elements $v_1, \ldots, v_n \in V$ and coefficients $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$ such that $x = \sum_{i=1}^n \alpha_i v_i$. Linearity of T implies that

$$y = T(x) = T\left(\sum_{i=1}^n \alpha_i v_i\right) = \sum_{i=1}^n \alpha_i T(v_i).$$

Since $T(v_1), \ldots T(v_n) \in T(V)$, this proves that $T(V)$ spans $\operatorname{Im} T$.

We next show that $T(V)$ is a linearly independent set. To see this let $u_1, \ldots, u_m$ be distinct vectors in $T(V)$, and suppose that the coefficients $\alpha_1, \ldots, \alpha_m \in \mathbb{F}$ are such that $\sum_{i=1}^m \alpha_i u_i = 0$. Note that $\operatorname{Ker} T = \{0\}$ implies that T is one-to-one, so since $u_1, \ldots, u_m$ are distinct vectors in $T(V)$, there exist distinct vectors $v_1, \ldots, v_m \in V$ such that $u_i = T(v_i)$ for $i = 1, \ldots, m$. Then, linearity of T implies that

$$T\left(\sum_{i=1}^m \alpha_i v_i\right) = \sum_{i=1}^m \alpha_i T(v_i) = \sum_{i=1}^m \alpha_i u_i = 0.$$

Since $\operatorname{Ker} T = \{0\}$, the above equality implies that $\sum_{i=1}^m \alpha_i v_i = 0$. Since V is a basis of X, vectors in V are linearly independent, implying that $\alpha_1 = \ldots = \alpha_m = 0$. This shows that vectors in U are linearly independent.

3. (20pts) Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$F_1(y_1, y_2) = y_2 + 1 \qquad F_2(y_1, y_2) = y_1^3 - 1$$

(a) Find the steady state y_s of the differential equation $y' = F(y)$.

Solution: Steady states are defined as the solution of the equation $F(y_s) = 0$. Therefore the unique steady state is $y_s = (1, -1)^T$.

(b) Linearize the differential equation around the steady state.

Solution: The Jacobian of F is given by

$$DF(y) = \begin{pmatrix} 0 & 1 \\ 3y_1^2 & 0 \end{pmatrix}$$

Evaluated at the steady state in part (a), this is

$$DF(y_s) = \begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix}$$

The linearized differential equation around y_s is $y' = DF(y_s)(y - y_s)$ where $DF(y_s)$ is the matrix given above.

(c) Find the general solution of the linear differential equation in (b).

Solution: The eigenvalues of of $DF(y_s)$ are given by the roots of the characteristic polynomial is $\lambda^2 - 3$, which are $\lambda_1 = \sqrt{3}$ and $\lambda_2 = -\sqrt{3}$. A pair of eigenvectors associated with these eigenvalues are given by $v_1 = (1, \sqrt{3})^T$ and $v_2 = (1, -\sqrt{3})^T$. Therefore, by Theorem 2 from lecture 14 notes, we know that the general solution of the linear differential equation in part (b) is given by:

$$y(t) = V \begin{pmatrix} e^{\sqrt{3}t} & 0 \\ 0 & e^{-\sqrt{3}t} \end{pmatrix} V^{-1}(y(0) - y_s) + y_s,$$

where V is the matrix whose columns are the eigenvectors v_1 and v_2, and V^{-1} is its inverse:

$$V = \begin{pmatrix} 1 & 1 \\ \sqrt{3} & -\sqrt{3} \end{pmatrix} \text{ and } V^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2\sqrt{3}} \\ \frac{1}{2} & -\frac{1}{2\sqrt{3}} \end{pmatrix}.$$

(d) Illustrate the dynamics of the linear differential equation in a phase-plane diagram.

Solution: Here you should draw the dynamics in $\mathbb{R}^2$ where the origin of the new coordinate system is y_s and the axes of the new parametrization are given by the eigenvectors v_1 and v_2. Since $\lambda_1 > 0$ the solution diverges along that v_1-axis, and since $\lambda_2 < 0$ the solution converges to y_s along the v_2-axis. Anywhere else the solution follows hyperbolic paths.

4. (15pts) For any continuous function $f : [0,1] \to \mathbb{R}$, define the functions $T(f) : [0,1] \to \mathbb{R}$ and $S(f) : [0,1] \to \mathbb{R}$ by

$$T(f)(x) = 1 + \int_0^x f(s)ds$$

$$S(f)(x) = \begin{cases} f(x + \frac{1}{2}) & \text{if } x < \frac{1}{2} \\ f(1) & \text{if } x \geq \frac{1}{2} \end{cases}$$

for every $x \in [0,1]$. Let $W(f) = \alpha T(f) + \beta S(f)$ for some $\alpha, \beta \in \mathbb{R}$. Show that if $|\alpha| + |\beta| < 1$, then there exists a continuous function $f : [0,1] \to \mathbb{R}$ such that $W(f) = f$.

Solution: Note that $T(f)$ is a continuous function since it is an integral. $S(f)$ is continuous on $[0, \frac{1}{2})$ because f is continuous; $S(f)$ is continuous on $(\frac{1}{2}, 1]$ because the constant function $f(1)$ is continuous; and $T(f)$ is continuous at $\frac{1}{2}$, because its left and right limits at $\frac{1}{2}$ exist and equal $f(1)$. Therefore, both $T(f)$ and $S(f)$ are also continuous functions from $[0,1]$ to $\mathbb{R}$. As a result, T, S, and W are operators on the Banach space $C([0,1])$ of continuous real-valued functions endowed with the sup-norm. We will prove that when $|\alpha| + |\beta| < 1$, then W is a contraction mapping with modulus $|\alpha| + |\beta|$ which will imply the desired result by the contraction mapping theorem.

Note first that for any $f, g \in C([0,1])$ and $x \in [0,1]$:

$$|T(f)(x) - T(g)(x)| = |\int_0^x [f(s) - g(s)]ds| \leq x\|f - g\|_\infty \leq \|f - g\|_\infty,$$

where the second inequality follows from $|f(s) - g(s)| \leq \|f - g\|_\infty$ for all $s \in [0,1]$. We therefore have

$$\|T(f) - T(g)\|_\infty = \sup\{|T(f)(x) - T(g)(x)| : x \in [0,1]\} \leq \|f - g\|_\infty.$$

Note also that for any $f, g \in C([0,1])$ and $x \in [0,1]$:

$$\begin{aligned} |S(f)(x) - S(g)(x)| &= \begin{cases} |f(x+\frac{1}{2}) - g(x+\frac{1}{2})| & \text{if } x < \frac{1}{2} \\ |f(1) - g(1)| & \text{if } x \geq \frac{1}{2} \end{cases} \\ &\leq \|f-g\|_\infty. \end{aligned}$$

We therefore also have

$$\|S(f) - S(g)\|_\infty = \sup\{|S(f)(x) - S(g)(x)| : x \in [0,1]\} \leq \|f-g\|_\infty.$$

Then, for any for any $f, g \in C([0,1])$:

$$\begin{aligned} \|W(f) - W(g)\|_\infty &= \|\alpha T(f) + \beta S(f) - [\alpha T(g) + \beta S(g)]\|_\infty \\ &= \|\alpha[T(f) - T(g)] + \beta[S(f) - S(g)]\|_\infty \\ &\leq |\alpha|\|[T(f) - T(g)]\|_\infty + |\beta|\|[S(f) - S(g)]\|_\infty \\ &\leq (|\alpha| + |\beta|)\|f-g)\|_\infty. \end{aligned}$$

Therefore, W is a contraction mapping with modulus $|\alpha| + |\beta| < 1$.

5. (20pts) Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function. Suppose that for every $x \in \mathbb{R}$, there exists $M_x > 0$ such that

$$\forall s \in l(0,x) \ \& \ k = 0, 1, 2, \ldots : \quad |f^{(k)}(s)| \leq M_x.$$

Show that for any $x \in X$

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k.$$[1]

Solution: Fix $x \in \mathbb{R}$, and consider the first version of Taylor's Theorem that we saw in class (Theorem 7 in lecture 1 slides) around the point 0:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k + \xi_{n,x}$$

where

$$\xi_{n,x} = f^{(n+1)}(0 + \lambda_{n,x} x) \frac{x^{n+1}}{(n+1)!} \text{ for some } \lambda_{n,x} \in (0,1).$$

[1] Remember the definitions $l(0,x) \equiv \{\alpha x : \alpha \in [0,1]\}$ and $\sum_{k=0}^{\infty} a_k \equiv \lim_{n\to\infty} \sum_{k=0}^{n} a_k$.

Rearranging the terms in the expansion, we have

$$\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k = f(x) - \xi_{n,x},$$

Therefore, if we can prove that $\xi_{n,x} \to 0$ as $n \to \infty$, we will have shown:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k = \lim_{n\to\infty} [f(x) - \xi_{n,x}] = f(x) - \lim_{n\to\infty} \xi_{n,x} = f(x).$$

To see that $\xi_{n,x} \to 0$ as $n \to \infty$, note that:

$$\begin{aligned} |\xi_{n,x}| &= |f^{(n+1)}(\lambda_{n,x} x) \frac{x^{n+1}}{(n+1)!}| \\ &= |f^{(n+1)}(\lambda_{n,x} x)| \frac{|x|^{n+1}}{(n+1)!} \\ &\leq M_x \frac{|x|^{n+1}}{(n+1)!} \end{aligned}$$

where the inequality above follows from the fact that $\lambda_{n,x} x \in l(0, x)$. Fix $N \in \mathbb{N}$ such that $N \geq 2x$, then for every $n \geq N$, we have:

$$\begin{aligned} |\xi_{n,x}| &\leq M_x \frac{|x|^{n+1}}{(n+1)!} \\ &\leq M_x \frac{|x|^N}{N!} \times \left[\frac{|x|}{N+1} \times \frac{|x|}{N+2} \times \ldots \times \frac{|x|}{n+1} \right] \\ &\leq M_x \frac{|x|^N}{N!} \left(\frac{1}{2} \right)^{n-N+1} \to 0 \end{aligned}$$

as $n \to \infty$.

6. (20pts) Let O be an open set in a metric space. Show that there exist countably many closed sets $C_1, C_2, C_3, \ldots$; such that $O = \cup_{k=1}^{\infty} C_k$. Feel free to take the fact stated in question 1 as given.

   Solution: Let (X, d) denote the metric space. For every $k \in \mathbb{N}$, define the set $D_k := \cup_{y \in X \setminus O} B_{\frac{1}{k}}(y)$. The set D_k is open since it is a union of open balls. Therefore, $C_k := X \setminus D_k$ is a closed set. Moreover $X \setminus O \subset D_k$ because for every $y \in X \setminus O$, we have $y \in B_{\frac{1}{k}}(y) \subset D_k$. This implies that $C_k \subset O$, so $\cup_{k=1}^{\infty} C_k \subset O$. To prove the reverse inclusion, take any $x \in O$. Since $x \notin X \setminus O$ and $X \setminus O$ is a closed set, we have by Question 1 that $d(x, X \setminus O) > 0$. Take any $K \in \mathbb{N}$

such that $\frac{1}{K} < d(x, X \setminus O)$. By the definition of $d(x, X \setminus O)$, for any $y \in X \setminus O$, $d(x, y) \geq d(x, X \setminus O) > \frac{1}{K}$, implying that $x \notin B_{\frac{1}{K}}(y)$. Therefore, $x \notin D_K$, so we have that $x \in C_K \subset \cup_{k=1}^{\infty} C_k$, proving the reverse inclusion.