Econ 204 (2014) - Final
08/20/2014

Instructions: This is a closed book exam. You have 3 hours. The weight of each question is indicated next to it. Write clearly, explain your answers, and be concise. You may use any result from class unless you are explicitly asked to prove it. Good luck!

1. (10pts) Suppose that A is a closed set in a metric space, and $x \notin A$. Show that
$$d(x, A) \equiv \inf\{d(x, y) : y \in A\} > 0.$$

2. (15pts) Let X and Y be (not necessarily finite dimensional) vector spaces over a field $\mathbb{F}$, and let $T \in L(X, Y)$ be such that $\operatorname{Ker} T = \{0\}$. Show that if V is a Hamel basis of X, then $T(V) \equiv \{T(v) : v \in V\}$ is a Hamel basis of $\operatorname{Im} T$.

3. (20pts) Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by
$$F_1(y_1, y_2) = y_2 + 1 \qquad F_2(y_1, y_2) = y_1^3 - 1$$
   (a) Find the steady state y_s of the differential equation $y' = F(y)$.
   (b) Linearize the differential equation around the steady state.
   (c) Find the general solution of the linear differential equation in (b).
   (d) Illustrate the dynamics of the linear differential equation in a phase-plane diagram.

4. (15pts) For any continuous function $f : [0, 1] \to \mathbb{R}$, define the functions $T(f) : [0, 1] \to \mathbb{R}$ and $S(f) : [0, 1] \to \mathbb{R}$ by
$$T(f)(x) = 1 + \int_0^x f(s)ds$$
$$S(f)(x) = \begin{cases} f(x + \frac{1}{2}) & \text{if } x < \frac{1}{2} \\ f(1) & \text{if } x \geq \frac{1}{2} \end{cases}$$
for every $x \in [0, 1]$. Let $W(f) = \alpha T(f) + \beta S(f)$ for some $\alpha, \beta \in \mathbb{R}$. Show that if $|\alpha| + |\beta| < 1$, then there exists a continuous function $f : [0, 1] \to \mathbb{R}$ such that $W(f) = f$.

5. (20pts) Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function. Suppose that for every $x \in \mathbb{R}$, there exists $M_x > 0$ such that

$$\forall s \in l(0, x) \ \& \ k = 0, 1, 2, \ldots : \quad |f^{(k)}(s)| \leq M_x.$$

Show that for any $x \in X$

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k.$$ [1]

6. (20pts) Let O be an open set in a metric space. Show that there exist countably many closed sets $C_1, C_2, C_3, \ldots$; such that $O = \cup_{k=1}^{\infty} C_k$. Feel free to take the fact stated in question 1 as given.

[1] Remember the definitions $l(0, x) \equiv \{\alpha x : \alpha \in [0, 1]\}$ and $\sum_{k=0}^{\infty} a_k \equiv \lim_{n\to\infty} \sum_{k=0}^{n} a_k$.