Economics 204
Fall 2012
Problem Set 1
Due Friday, July 27 in Lecture

1. Use induction to prove the following statements.
   (a) The equality $\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2$ holds for all $n \in \mathbb{N}$;
   (b) The inequality $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \geq \sqrt{n}$ holds for all $n \in \mathbb{N}$;
   (c) The inequality $(1+x)^n \geq 1 + nx$ holds for all $n \in \mathbb{N}$ and all $x \in [-1, \infty)$.

2. Let A and B be subsets of $\mathbb{R}$ such that their complements are countably infinite. Prove $A \cap B \neq \varnothing$.

3. Prove that there are uncountably many infinite subsets (i.e. subsets with infinitely many elements) of $\mathbb{N}$. (If you need to, you can use the fact that the countable union of countable sets is countable.)

4. A collection $\mathcal{S}$ of subsets of some fixed set X which has the properties
   - $\varnothing \in \mathcal{S}$;
   - $A, B \in \mathcal{S} \Rightarrow A \cap B \in \mathcal{S}$;
   - $A, B \in \mathcal{S}, A \subseteq B \Rightarrow B \setminus A = \bigcup_{k=1}^{n} A_k$ for some pairwise disjoint sets $A_1, \ldots, A_n \in \mathcal{S}$

   is called a *semiring*.[1,2]

   Let $X = Y \times Z$ and let $\mathcal{A}$ and $\mathcal{B}$ be semirings of some sets Y and Z, respectively. Let $\mathcal{S} = \{A \times B : A \in \mathcal{A}, B \in \mathcal{B}\}$. Prove that $\mathcal{S}$ is a semiring of the set X.

5. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a function with the following properties:
   - $f(x) = 0$ if and only if $x = 0$;

[1] $B \setminus A$ is the set difference of B and A, denoted by $B \sim A$ in de la Fuente. More specifically, $B \setminus A = \{x \in X : x \in B, x \notin A\}$.

[2] For example, the collection of all intervals on the real line of the form $[a,b], [a,b), (a,b], (a,b)$ is a semiring, where $[a,a] = \{a\}$.

- f is non-decreasing (i.e. $x \geq y \Rightarrow f(x) \geq f(y)$);
- $f(x+y) \leq f(x) + f(y)$ for all $x, y \geq 0$.

Show that if (X, d) is a metric space, then $(X, f \circ d)$ is also a metric space.

6. Let (X, d) be a metric space and let $\{x_n\}$ and $\{y_n\}$ be sequences in X that converge to x and y respectively.

    (a) Prove that the sequence $\{d(x_n, y_n)\}$ converges to $d(x, y)$.

    (b) Let $X = \mathbb{R}$ and d be the usual metric on $\mathbb{R}$. Define $z_n = \max\{x_n, y_n\}$ for all $n \in \mathbb{N}$. Prove that the sequence $\{z_n\}$ converges to $\max\{x, y\}$.