Economics 204
Fall 2013
Problem Set 1
Due Friday, August 2 in Lecture

1. A number of students met to discuss 204 homework. Some of them shook each other hands. Prove that the number of students who shook others' hands odd number of times is, in fact, even.

2. Determine whether this formula is always right or sometimes wrong. Prove it if it is right. Otherwise, give a couter-example and state (and prove) the right formula.
$$A \cap (B \setminus C) = (A \cap B) \setminus C = (A \cap B) \setminus (A \cap C)$$

3. Certain subsets of a given set S are called A-sets and others are called B-sets. Suppose that these subsets are chosen in such a way that the following properties are satisfied:
   - The union of any collection of A-sets is and A-set.
   - The intersection of any finite number of A-sets is an A-set.
   - The complement of an A-set is a B-set and the complement of a B-set is an A-set.

   Prove directly, without appealing to some general result, the following:
   (a) The intersection of any collection of B-sets is a B-set.
   (b) The union of any finite number of B-sets is a B-set.

4. Are there $a, b \in \mathbf{R} \setminus \mathbf{Q}$ such that
   (a) $a + b \in \mathbf{Q}$
   (b) $a \cdot b \in \mathbf{Q}$
   (c) $a^b \in \mathbf{Q}$

   Prove your assertions.

5. Call a sequence of real numbers $x = \{x_n\}$ finite if there exists $N \in \mathbf{N}$ such that $x_n = 0$ for all $n > N$. Let set S consists of all finite sequences that are constructed from some countable set X. Prove that S is countable.

6. Let A and B be two sets of positive numbers bounded above, and let $\alpha = \sup A$ and $\beta = \sup B$. Let C be the set of all products of the form $a \cdot b$, where $a \in A$ and $b \in B$. Prove that $\alpha \cdot \beta = \sup C$.

7. Let X be any nonempty set. If a distance function $d : X \times X \to \mathbf{R}_+$ that satisfies assumptions of *(i)* symmetry, *(ii)* triangle inequality and *(iii)* $d(x, x) = 0$ for all $x \in X$, then we say that d is a *semi-metric* on X, and $(X,\, d)$ is a *semi-metric* space.
For any semi-metric space X, define the binary relation $\approx$ on X by $x \approx y$ iff $d(x,\, y) = 0$. Now, define $[x] = \{\, y \in Y : x \approx y\}$ for all $x \in X$, and let $\mathcal{X} = \{[x] : x \in X\}$. Finally, define $D : \mathcal{X} \times \mathcal{X} \to \mathbf{R}_+$ by $D([x], [y]) = d(x, y)$.

   (a) Show that $\approx$ is an equivalence relation on X.

   (b) Prove that $(\mathcal{X}, D)$ is a metric space.

8. Let c_{00} be the space of all finite sequences of real numbers.

   (a) Show that for $p \in [1, +\infty)$ the real-valued operation $\| \cdot \|_p$

$$\|x\|_p = \left(\sum_{n=0}^{\infty} |x_n|^p \right)^{\frac{1}{p}}$$

   is a norm.

   (b) Are $|| \cdot ||_p$ equivalent to $|| \cdot ||_q$ on c_{00} for $p \neq q$?