Econ 204 – Problem Set 1

Due Friday, August 1

1. Let A, B, C be sets. Prove the following statements:

   (a) $C \setminus (A \cup B) = (C \setminus A) \cap (C \setminus B)$

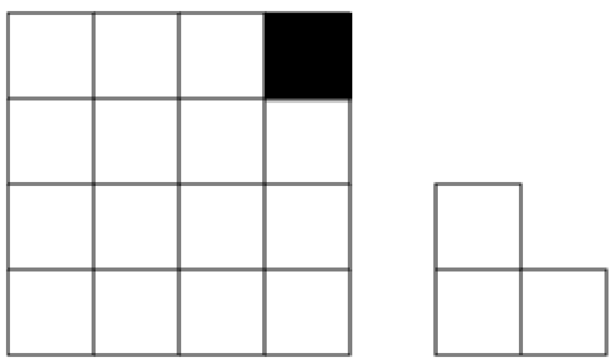
   (b) $C \setminus (A \cap B) = (C \setminus A) \cup (C \setminus B)$

2. Use the principle of mathematical induction to prove the following statements:

   (a) A set S with n elements has 2^n subsets. (note: do not forget about the empty set)

   (b) $\left|\sum_{n=1}^{N} x_n\right| \leq \sum_{n=1}^{N} |x_n|$, for $x_n \in \mathbb{R}$

   (c) Prove that any grid made up of $2^n \times 2^n$ tiles can be covered except for one corner tile by L-shaped triominoes (the triominoes may rotated). The figure below shows an example of a 4×4 grid (left) where all of the non-shaded tiles must be covered by a triomino (right). Note: Visual proofs of the base and inductive steps are fine.



3. Let A and B be subsets of any uncountable set X such that their complements are countably infinite. Prove that $A \cap B \neq \varnothing$.

4. If $x \neq 0$ is rational and y is irrational, prove that $x + y$ and $x \cdot y$ are irrational. If $x \neq 0$ is instead irrational, does the statement still hold?

5. Recall the definition of an *ordered field*: a field F with a binary relation "$\leq$" such that $\forall x, y, z \in F$, we have:

   - Totality: $x \leq y$ or $y \leq x$
   - Antisymmetry: $x \leq y$ and $y \leq x \implies x = y$
   - Transitivity: $x \leq y$ and $y \leq z \implies x \leq z$

- The order complies with addition and multiplication: $y \leq z \implies x + y \leq x + z$ and $x \geq 0, y \geq 0 \implies x \cdot y \geq 0$

We define "$x < y$" as "$x \leq y$" but not "$y \leq x$"; similarly for $x > y$.

(a) Prove the following properties of any ordered field:

i. $x \geq 0 \implies -x \leq 0$ and vice versa.

ii. $x \geq 0$ and $y \leq z \implies x \cdot y \leq x \cdot z$

iii. $x \leq 0$ and $y \leq z \implies x \cdot y \geq x \cdot z$

iv. $x \neq 0 \implies x^2 > 0$

v. $0 < x < y \implies 0 < y^{-1} < x^{-1}$

(b) Using the above properties, prove that the complex field $\mathbb{C}$ cannot be made into an ordered field.

6. Let A be a subet of $\mathbb{R}$ that is nonempty and bounded below. Define the set $-A = \{-a : a \in A\}$. Prove that $\inf A = -\sup(-A)$.

7. Define the following distance function on the set of real numbers:

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

(a) Prove that $(\mathbb{R}, d)$ is a metric space.

(b) Identify the open (and closed) balls in the topology induced by this metric.