Economics 204
Fall 2012
Problem Set 1 Suggested Solutions

1. Use induction to prove the following statements.

   (a) The equality $\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2$ holds for all $n \in \mathbb{N}$;

   (b) The inequality $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \geq \sqrt{n}$ holds for all $n \in \mathbb{N}$;

   (c) The inequality $(1+x)^n \geq 1 + nx$ holds for all $n \in \mathbb{N}$ and all $x \in [-1, \infty)$.

Solution:

(a) The **base step** $n = 1$ is straightforward - both sides of the equality are equal to 1.

Induction step: Assume $\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2$ holds for some $n \in \mathbb{N}$. Now consider the corresponding equality for $n+1$. Starting from the right-hand side, we have:

$$
\begin{aligned}
\left(\sum_{i=1}^{n+1} i\right)^2 &= \left(\sum_{i=1}^{n} i + (n+1)\right)^2 \\
&= \left(\sum_{i=1}^{n} i\right)^2 + 2\left(\sum_{i=1}^{n} i\right)(n+1) + (n+1)^2 \\
&= \sum_{i=1}^{n} i^3 + 2\frac{(n+1)n}{2}(n+1) + (n+1)^2 \\
&= \sum_{i=1}^{n} i^3 + (n+1)^2 n + (n+1)^2 \\
&= \sum_{i=1}^{n} i^3 + (n+1)^2(n+1) \\
&= \sum_{i=1}^{n} i^3 + (n+1)^3 \\
&= \sum_{i=1}^{n+1} i^3,
\end{aligned}
$$

where the third equality follows from the induction hypothesis and from the fact that $\sum_{i=1}^{n} i = \frac{(n+1)n}{2}$. So by mathematical induction, $\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2$ for all $n \in \mathbb{N}$.

(b) **Base step $n = 1$:** both sides are equal to 1 ($1/\sqrt{1} = 1$ and $\sqrt{1} = 1$) and, obviously, $1 \geq 1$.

Induction step: Assume $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \geq \sqrt{n}$ holds for some $n \in \mathbb{N}$. Now consider the left-hand side of the inequality for $n+1$:

$$
\begin{aligned}
\sum_{i=1}^{n} \frac{1}{\sqrt{i}} + \frac{1}{\sqrt{n+1}} \geq \sqrt{n} + \frac{1}{\sqrt{n+1}} &= \frac{\sqrt{n(n+1)} + 1}{\sqrt{n+1}} \\
\geq \frac{\sqrt{n^2} + 1}{\sqrt{n+1}} = \frac{n+1}{\sqrt{n+1}} &= \sqrt{n+1},
\end{aligned}
$$

where the first inequality follows from the induction hypothesis. So by mathematical induction, $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \geq \sqrt{n}$ for all $n \in \mathbb{N}$.

(c) **Base step $n = 1$:** both sides are equal to $1 + x$.

Induction step: Fix $x \geq -1$ and assume that $(1+x)^n \geq 1+nx$ holds for some $n \in \mathbb{N}$. Now consider the left-hand side of the inequality for $n+1$:

$$
\begin{aligned}
(1+x)^{n+1} = (1+x)^n(1+x) &\geq (1+nx)(1+x) \\
&= 1 + nx + x + nx^2 \\
&\geq 1 + (n+1)x,
\end{aligned}
$$

where the first inequality follows from the induction hypothesis and the fact that $1 + x \geq 0$, while the second inequality follows from the fact that $nx^2 \geq 0$. So by mathematical induction, $(1 + x)^n \geq 1 + nx$ for all $n \in \mathbb{N}$ and $x \in [-1, \infty)$.

2. Let A and B be subsets of $\mathbb{R}$ such that their complements are countably infinite. Prove $A \cap B \neq \varnothing$.

   Solution: Since $\mathbb{R}$ is uncountably infinite, the sets A and B must also be uncountably infinite. To see that, note that if, say, the set A were finite or countably infinite, then $\mathbb{R} = A \cup A^C$ would be the union of two sets that are at most countably infinite and thus would be countably infinite itself.

   Now, toward contradiction, assume that $A \cap B = \varnothing$. This is equivalent to saying $A \subseteq B^C$. This is a contradiction since A is uncountably infinite, while B's complement B^C is countably infinite, and an uncountably infinite set cannot be contained in a countably infinite one.

3. Prove that there are uncountably many infinite subsets (i.e. subsets with infinitely many elements) of $\mathbb{N}$. (If you need to, you can use the fact that the countable union of countable sets is countable.)

   Solution: In class we showed that $\mathbb{N}$ has uncountably many subsets. Following the logic of the previous problem, it suffices to show that there are countably many finite subsets of $\mathbb{N}$.

   Before continuing, let us quickly prove a useful auxiliary result. Namely, we'll show using induction that $\mathbb{N}^k$ is countable for all $k \in \mathbb{N}$ (i.e. the

k-fold Cartesian product of $\mathbb{N}$ has the same cardinality as $\mathbb{N}$). The **base step** $k = 1$ is straightforward since $\mathbb{N}^1 = \mathbb{N}$ is countable by definition. For the **induction step**, assume that $\mathbb{N}^k$ is countable and consider $\mathbb{N}^{k+1} = \mathbb{N}^k \times \mathbb{N} = \{(x, y) : x \in \mathbb{N}^k, y \in \mathbb{N}\}$. Since both $\mathbb{N}^k$ and $\mathbb{N}$ are countable, $\mathbb{N}^{k+1}$ is numerically equivalent to $\mathbb{Q} = \left\{\frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{N}\right\}$, which is countable as we established in class.

Now fix some $k \in \mathbb{N}$ and consider all subsets of $\mathbb{N}$ with cardinality k - let's call those $\mathcal{P}^k(\mathbb{N})$. Note that there exist obvious mappings that embed $\mathcal{P}^k(\mathbb{N})$ as a subset of $\mathbb{N}^k$.[1] Thus the cardinality of $\mathcal{P}^k(\mathbb{N})$ is no larger than the cardinality of $\mathbb{N}^k$. Hence $\mathcal{P}^k(\mathbb{N})$ is countably infinite.[2]

But all finite subsets of $\mathbb{N}$ are just $\bigcup_{k\in\mathbb{N}} \mathcal{P}^k(\mathbb{N})$ and, as a countable union of countable sets, it is countable. As noted above, this suffices to prove the desired result.

4. A collection $\mathcal{S}$ of subsets of some fixed set X which has the properties

   - $\varnothing \in \mathcal{S}$;
   - $A, B \in \mathcal{S} \Rightarrow A \cap B \in \mathcal{S}$;
   - $A, B \in \mathcal{S}, A \subseteq B \Rightarrow B \setminus A = \bigcup_{k=1}^n A_k$ for some pairwise disjoint sets $A_1, \ldots, A_n \in \mathcal{S}$

   is called a *semiring*.[3,4]

   Let $X = Y \times Z$ and let $\mathcal{A}$ and $\mathcal{B}$ be semirings of some sets Y and Z, respectively. Let $\mathcal{S} = \{A \times B : A \in \mathcal{A}, B \in \mathcal{B}\}$. Prove that $\mathcal{S}$ is a semiring of the set X.

   Solution: First, $\varnothing = \varnothing \times \varnothing \in \mathcal{S}$ since the empty set is both in $\mathcal{A}$ and $\mathcal{B}$.

   For intersections, we have

[1] For example, the function $f : \mathcal{P}^k(\mathbb{N}) \to \mathbb{N}^k$ defined by $f(\{a_1, \ldots, a_k\}) = (a_1, \ldots, a_k)$ would do just fine here.

[2] It is indeed infinite since it contains all sets of the form $\{n, n+1, \ldots, n+k-1\}$ for all $n \in \mathbb{N}$.

[3] $B \setminus A$ is the set difference of B and A, denoted by $B \sim A$ in de la Fuente. More specifically, $B \setminus A = \{x \in X : x \in B, x \notin A\}$.

[4] For example, the collection of all intervals on the real line of the form $[a, b], [a, b), (a, b], (a, b)$ for all $a, b \in \mathbb{R}$ is a semiring (where $[a, a] = \{a\}$).

$$(A \times B) \cap (E \times F) = (A \cap E) \times (B \cap F)$$

for any $A, E \in \mathcal{A}$ and $B, F \in \mathcal{B}$. To see why the set equality above holds, note that if $(a, b) \in (A \times B) \cap (E \times F)$, then it follows that $a \in A \cap E$ and $b \in B \cap F$ and therefore $(a, b) \in (A \cap E) \times (B \cap F)$. This establishes the inclusion

$$(A \times B) \cap (E \times F) \subseteq (A \cap E) \times (B \cap F).$$

The other inclusion is also straightforward: if $(a, b) \in (A \cap E) \times (B \cap F)$, then $a \in A$, $a \in E$, and, similarly, $b \in B$ and $b \in F$. Thus $(a, b) \in A \times B$ and $(a, b) \in E \times F$. Hence (a, b) is also in the intersection of these two Cartesian products.

Since $A \cap E \in \mathcal{A}$ and $B \cap F \in \mathcal{B}$ by the second property of semirings, then $(A \cap E) \times (B \cap F) \in \mathcal{S}$ and so $(A \times B) \cap (E \times F) \in \mathcal{S}$.

For differences, choose sets $A, E \in \mathcal{A}$ and $B, F \in \mathcal{B}$ such that $E \times F \subseteq A \times B$ or, equivalently, $E \subseteq A$ and $F \subseteq B$. Now consider the difference

$$D = (A \times B) \setminus (E \times F) = ((A \setminus E) \times B) \cup (E \times (B \setminus F)).$$

To see that the set equality indeed holds, let $(a, b) \in (A \times B) \setminus (E \times F)$. This means that $(a, b) \in A \times B$ but either $a \notin E$ or $b \notin F$ (or both). If $a \notin E$ then $(a, b) \in (A \setminus E) \times B$. If $a \in E$ but $b \notin F$ then $(a, b) \in E \times (B \setminus F)$. Thus $(a, b) \in ((A \setminus E) \times B) \cup (E \times (B \setminus F))$, which establishes the inclusion of the LHS of the equality in its RHS.

Conversely, if $(a, b) \in ((A \setminus E) \times B) \cup (E \times (B \setminus F))$ then it is clear that $(a, b) \in A \times B$. At least one of two things is also true: either $a \in A \setminus E$ and hence $a \notin E$ or $b \in B \setminus F$ and hence $b \notin F$. Either way, this means $(a, b) \notin E \times F$ and hence $(a, b) \in (A \times B) \setminus (E \times F)$, which establishes the other inclusion and hence the equality.

As a next step, we can express the set difference D as follows

$$\begin{aligned}
D &= ((A \setminus E) \times B) \cup (E \times (B \setminus F)) \\
&= \left(\left(\bigcup_{1 \le j \le m} G_j\right) \times B\right) \cup \left(E \times \left(\bigcup_{1 \le k \le n} H_k\right)\right) \\
&= \bigcup_{1 \le j \le m} (G_j \times B) \cup \bigcup_{1 \le k \le n} (E \times H_k)
\end{aligned}$$

for some $m, n \in \mathbb{N}$, pairwise disjoint sets $G_j \in \mathcal{A}$, and disjoint $H_k \in \mathcal{B}$. Note that we can express the set difference $A \setminus E$ in this manner using the third property of semirings because of the fact that A and E are sets in the semiring $\mathcal{A}$. (Similarly for $B \setminus F$.)

Since the G_j sets are pairwise disjoint, so are the sets of the form $G_j \times B$. By the way we've defined the G_j sets, we also have $G_j \cap E = \varnothing$. So the $G_j \times B$ sets and the $E \times H_k$ sets are pairwise disjoint. It is clear that these sets also belong to $\mathcal{S}$. This establishes that $\mathcal{S}$ satisfies the third property of semirings and completes the proof.

5. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a function with the following properties:

   - $f(x) = 0$ if and only if $x = 0$;
   - f is non-decreasing (i.e. $x \geq y \Rightarrow f(x) \geq f(y)$);
   - $f(x + y) \leq f(x) + f(y)$ for all $x, y \geq 0$.

   Show that if (X, d) is a metric space, then $(X, f \circ d)$ is also a metric space.

   Solution: To check that $(X, f \circ d)$ is indeed a metric space, we need to verify that $f \circ d$ satisfies the three properties of a metric.

   1. Since f maps into $\mathbb{R}_+$, we have $(f \circ d)(x, y) \geq 0$ for all $x, y \in X$. Since $d(x, y) = 0$ iff $x = y$ and $f(z) = 0$ iff $z = 0$, this implies $(f \circ d)(x, y) = 0$ iff $x = y$.
   2. Since $d(x, y) = d(y, x)$ for all $x, y \in X$, we have $(f \circ d)(x, y) = (f \circ d)(y, x)$.
   3. Fix $x, y, z \in X$. Then

   $$\begin{aligned}(f \circ d)(x, z) &= f(d(x, z)) \\ &\leq f(d(x, y) + d(y, z)) \\ &\leq f(d(x, y)) + f(d(y, z)) \\ &= (f \circ d)(x, y) + (f \circ d)(y, z),\end{aligned}$$

   where the first inequality follows from the fact that d satisfies the triangle inequality (as a metric) and f is non-decreasing by assumption, while the second inequality follows from f's third property.

Hence $f \circ d$ satisfies the triangle inequality, which completes the proof.

6. Let (X, d) be a metric space and let $\{x_n\}$ and $\{y_n\}$ be sequences in X that converge to x and y respectively.

   (a) Prove that the sequence $\{d(x_n, y_n)\}$ converges to $d(x, y)$.

   (b) Let $X = \mathbb{R}$ and d be the usual metric on $\mathbb{R}$. Define $z_n = \max\{x_n, y_n\}$ for all $n \in \mathbb{N}$. Prove that the sequence $\{z_n\}$ converges to $\max\{x, y\}$.

Solution:

(a) Fix $\varepsilon > 0$. Since $\{x_n\}$ converges to x, we can find some $N_x \in \mathbb{N}$ such that for all $n \geq N_x$ we have $d(x_n, x) < \varepsilon/2$. Similarly, let $N_y \in \mathbb{N}$ be such that for all $n \geq N_y$ we have $d(y_n, y) < \varepsilon/2$. Let $N = \max\{N_x, N_y\}$. Then the two inequalities above hold for all $n \geq N$. Using the triangle inequality we get

$$\begin{aligned} d(x_n, y_n) &\leq d(x_n, x) + d(x, y) + d(y, y_n) \\ &< \varepsilon/2 + d(x, y) + \varepsilon/2 = d(x, y) + \varepsilon \\ &\Rightarrow d(x_n, y_n) - d(x, y) < \varepsilon \end{aligned}$$

for all $n \geq N$. Similarly:

$$\begin{aligned} d(x, y) &\leq d(x, x_n) + d(x_n, y_n) + d(y_n, y) \\ &< \varepsilon/2 + d(x_n, y_n) + \varepsilon/2 = d(x_n, y_n) + \varepsilon \\ &\Rightarrow d(x, y) - d(x_n, y_n) < \varepsilon \end{aligned}$$

for all $n \geq N$. Combining the two inequalities, we get

$$|d(x, y) - d(x_n, y_n)| < \varepsilon$$

for all $n \geq N$. Since ε was chosen arbitrarily, $\{d(x_n, y_n)\}$ converges to $d(x, y)$.

(b) We consider two cases separately. First, assume that $x = y = \max\{x, y\}$. Then for any $\varepsilon > 0$ we have some $N_x, N_y \in \mathbb{N}$ such that for all $n \geq N_x$ we have $|x_n - x| = |x_n - \max\{x, y\}| < \varepsilon$ and for all $m \geq N_y$ we have $|y_m - y| = |y_m - \max\{x, y\}| < \varepsilon$. Therefore for all $n \geq \max\{N_x, N_y\}$ we have $|z_n - \max\{x, y\}| < \varepsilon$. Since ε was chosen arbitrarily, $\{z_n\}$ converges to $\max\{x, y\}$.

Now consider the case $x \neq y$. Assume without loss of generality that $x > y$. Fix $\varepsilon = \frac{x-y}{2}$ and notice that $x - \varepsilon = y + \varepsilon$. There are some $N_x, N_y \in \mathbb{N}$ such that for all $n \geq N_x$ we have $x_n \in (x - \varepsilon, x + \varepsilon)$ and for all $m \geq N_y$ we have $y_m \in (y - \varepsilon, y + \varepsilon) = (y - \varepsilon, x - \varepsilon)$. Hence for all $n \geq \max\{N_x, N_y\}$ we have $x_n > y_n$ and hence $z_n = x_n$. So the tail of the sequence $\{z_n\}$ coincides with the tail of the sequence $\{x_n\}$ and therefore they both converge to $x = \max\{x, y\}$.