Economics 204
Fall 2013
Problem Set 1 Suggested Solutions

1. A number of students met to discuss 204 homework. Some of them shook each other hands. Prove that the number of students who shook others' hands odd number of times is, in fact, even.

   Solution. To begin, lets assume without any loss of generality that students shook hands in turn rather then all at once. Also, lets call "odd" and "even" students as those that made an "odd" or "even" number of handshakes.

   Consider the base case. Note that when $n = 0$ or $n = 1$ (either no students or only one student came to a meeting) no handshake took place and 0 is an even number. Also, after first handshake its both participants became "odd," (i.e. they just made one handshake) and two is clearly an even number.

   Now, lets make an induction hypothesis — n students shook others' hands odd number of times and n is even. When we consider the handshakes that involve $(n + 1)$–th student, we have a number of possibilities. Clearly, new handshakes could be made between "old" students and between "old" students and a newcomer (note that since newcomer made no handshakes she or he is an "even" student). Consider three possible cases:

   Case 1. Handshake between two "even" students. After such handshake each of them becomes an "odd" one, thus overall number of "odd" students goes up by two.

   Case 2. Handshake between two "odd" students. After such handshake each of them becomes an "even" one, thus overall number of "odd" students decreases by two.

   Case 3. Handshake between "odd" and "even" students. Observe that in this case, "odd" student becomes and "even" one and vise versa. Thus, the overall number of "odd" students is unaffected. We are done.

   As we can see, in each case the number of "odd" students remain even. We have proven an induction hypothesis and, thus, we are done.

2. Determine whether this formula is always right or sometimes wrong. Prove it if it is right. Otherwise, give a couter-example and state (and prove) the right formula.
$$A \cap (B \setminus C) = (A \cap B) \setminus C = (A \cap B) \setminus (A \cap C)$$

   Solution. This expression is correct. First, lets us prove that $A \cap (B \setminus C) \subset (A \cap B) \setminus (A \cap C)$. So, fix $x \in A \cap (B \setminus C)$. Thus, $x \in A$ and $x \in B \setminus C$. Therefore, $x \in A$ and $x \in B$ and $x \notin C$. Two former statements yield $x \in A \cap B$. Besides $x \notin C$

implies that $x \notin A \cap C$. So, $x \in A \cap B$ and $x \notin A \cap C \Longrightarrow x \in (A \cap B) \setminus (A \cap C)$. So, we have proved that $A \cap (B \setminus C) \subset (A \cap B) \setminus (A \cap C)$. Now, lets prove that $(A \cap B) \setminus C \subset A \cap (B \setminus C)$. We have $x \in A$ and $x \in B$ and $x \notin C$. Therefore, $x \in A$ and $x \in B \setminus C$. Thus, $x \in A \cap (B \setminus C)$.

Finally, lets prove that $(A \cap B) \setminus (A \cap C) \subset (A \cap B) \setminus C$. Let $x \in (A \cap B) \setminus (A \cap C)$. Then we have that $x \in A \cap B$ and $x \notin A \cap C$, therefore, since $x \in A \cap B$, we have that $x \in A$ and $x \in B$. $x \notin A \cap C$ implies that $x \notin C$. Thus, we have that $x \in A \cap B$ and $x \notin C$ which means that $x \in (A \cap B) \setminus C$. Therefore, we have proved that

$$(A \cap B) \setminus (A \cap C) \subset (A \cap B) \setminus C \subset A \cap (B \setminus C) \subset (A \cap B) \setminus (A \cap C).$$

Clearly this implies that $A \cap (B \setminus C) = (A \cap B) \setminus C = (A \cap B) \setminus (A \cap C)$. We are done.

3. Certain subsets of a given set S are called A-sets and others are called B-sets. Suppose that these subsets are chosen in such a way that the following properties are satisfied:

   - The union of any collection of A-sets is and A-set.
   - The intersection of any finite number of A-sets is an A-set.
   - The complement of an A-set is a B-set and the complement of a B-set is an A-set.

   Prove directly, without appealing to some general result, the following:

   (a) The intersection of any collection of B-sets is a B-set.

   (b) The union of any finite number of B-sets is a B-set.

   Solution.
   (a) Let I index B_I, a collection of B-sets, and let $x \in \bigcap_{i \in I} B_i$. Now $x \in \bigcap_{i \in I} B_i$ if and only if $x \in B_i \, \forall \, i \in I$, which is true if and only if $x \notin B_i^c$ for all $i \in I$, which is true if and only if $x \notin \bigcup_{i \in I} B_i^c$. This means that $\bigcap_{i \in I} B_i = (\bigcup_{i \in I} B_I^c)^c$. Because B_i is a B-set for all i, the complement B_i^c is an A-set. The union of these A-sets is an A-set, and its complement is, in turn, a B-set. Thus $\bigcap_{i \in I} B_i$ is a B-set.

   (b) Now let I be finite. $x \in \bigcup_{i \in I} B_i \iff x \in B_i$ for some $i \in I$. This is true if and only if $x \notin B_i^c$ for some $i \in I$ which is true if and only if $x \notin \bigcap_{i \in I} B_i^c$. Thus, $\bigcup_{i \in I} B_i = (\bigcap_{i \in I} B_i^c)^c$. B_i^c is an A-set for all i so the intersection of B_i is also an A-set and the complement of this intersection is a B-set. Thus, $\bigcup_{i \in I} B_i$ is a B-set.

4. Are there a, $b \in \mathbf{R} \setminus \mathbf{Q}$ such that

   (a) $a + b \in \mathbf{Q}$

   (b) $a \cdot b \in \mathbf{Q}$

   (c) $a^b \in \mathbf{Q}$

   If you claim that such a and b exist, please give an example, if not — prove your assertion.

   Solution. The answer is yes in all three cases. For sum, take $a = 1 - \sqrt{2}$ and $b = 1 + \sqrt{2}$, and for product, take $a = 2\sqrt{3}/3$ and $b = \sqrt{3}$. Now, for power consider the following two pairs of numbers $b_1 = -1 - \sqrt{5}$, $b_2 = -1 + \sqrt{5}$ and $a_1 = 2^{1/b_1}$, $a_2 = 2^{1/b_2}$. We have $a_1^{b_1} = a_2^{b_2} = 2$.
   Clearly, in all those cases a, $b \in \mathbf{R} \setminus \mathbf{Q}$, for sum and product it is immediate. It just remains to convince ourselves that a_1 and a_2 are in $\mathbf{R} \setminus \mathbf{Q}$. Notice, that at least one of those numbers must be irrational, since if both were rationals, we will not be able to obtain an equality

$$a_1 a_2 = \sqrt{2}.$$

   We are done.

5. Call a sequence $x = \{ x_n \}$ finite if there exists $N \in \mathbf{N}$ such that $x_n = 0$ for all $n > N$. Let set S consists of all finite sequences that are constructed from some countable set X. Prove that S is countable.

   Solution. Lets denote by "length" of a finite sequence that $N \in \mathbf{N}$ such that $x_n = 0$ for all $n > N$ and $x_{n-1} \neq 0$. Let A_n be the set of all finite sequences with lenght n and let A_0 be set containing an empty sequence. Clearly, $S = \bigcup_{n=0}^{\infty} A_n$. Note that if X is finite then each A_n will have at most finite number of elements (more precisely, it would be $|X|$ choose n, where $|X|$ is the cardinality of X.) Thus, if X is finite we are done because S is clearly countable for instance by "Hilbert hotel" argument. Now, if S is not finite, then we will get the result we desire if we can prove two facts: first, that each A_n is countable for every $n \in \mathbf{N}$, and, second, that a countable union of countable sets is countable.

   Before we do so, lets prove two small lemmas.

   Lemma 1. If $f : A \to \mathbf{N}$ is an injection, then A must be countable.
   Proof. Consider the set $n(A) = \{n \in \mathbf{N} : \exists a \in A \text{ such that } f(a) = n\}$. If $n(A)$ is finite then then we are done, so suppose it is infinite. We must show it is countable.

$$\begin{aligned}
\text{set } n_0 &= \min\{n \in n(A)\} \\
\text{set } n_1 &= \min\{n \in n(A) \setminus \{n_0\}\} \\
\text{set } n_2 &= \min\{n \in n(A) \setminus \{n_0, n_1\}\} \\
&\vdots \\
\text{set } n_k &= \min\{n \in n(A) \setminus \{n_0, n_1, \ldots, n_{k-1}\}\}
\end{aligned}$$

Since $n(A)$ is infinite and f is $1-1$, $n_1 < n_2 < \ldots$ and n_k is well-defined for each k.

Lemma 2. The n-fold Cartesian product of $\mathbf{N}$ has the same cardinality as $\mathbf{N}$, i.e. it is countable.

Proof. We show it by induction. Base step $n = 1$ is straightforward since $\mathbf{N}^1 = \mathbf{N}$ is countable by definition. For the induction step, assume that $\mathbf{N}^n$ is countable and consider $\mathbf{N}^{n+1} = \mathbf{N}^n \times \mathbf{N} = \{(x, y) : x \in \mathbf{N}^n, y \in \mathbf{N}\}$. Since both $\mathbf{N}^n$ and $\mathbf{N}$ are countable, $\mathbf{N}^{n+1}$ is numerically equivalent to $\mathbf{Q} = \left\{\frac{m}{n} : m \in \mathbf{N}, n \in \mathbf{N}\right\}$, which is countable as we established in class.

Lets start with the our first claim. Observe that since S is countable there is a bijection between set of natural numbers $\mathbf{N}$ and S, thus, without any loss of generality we can consider finite sequences of natural numbers. Having made this observation, we immediately see that the structure of A_n sets is quite simple because any finite sequence of natural numbers of length n can represented as a subset of $\mathbf{N}^n$. If so, there exists an obvious mappings that embed all finite sequences of $\mathbf{N}$ of length n as a subset of $\mathbf{N}^n$. An example of such mappings would be the function $f(\{k_1, \ldots, k_n\}) = (k_1, \ldots, k_n)$, which is clearly injective. Thus, by Lemma 1 and 2 A_n is countable and we can move on to proving our second claim.

To show that a countable union of countable sets is countable, note that without any loss of generality we can suppose that sets are all pairwise disjoint (in our case, all A_n are by construction). Because each A_n is countably infinite, we can enumerate them as $A_n = \{a_n^1, a_n^2, \ldots,\}$. Again, we define a injective mapping $f : A_n \to \mathbf{N}^2$ by $f(a_n^k) = (n, k)$. By Lemma 1 and 2, $\bigcup_{n=1}^{\infty} A_n$ is countable.

6. Let A and B be two sets of positive real numbers bounded above, and let $\alpha = \sup A$ and $\beta = \sup B$. Let C be the set of all products of the form $a \cdot b$, where $a \in A$ and $b \in B$. Prove that $\alpha \cdot \beta = \sup C$.

   Solution. Let $C = \{\, a \cdot b \,|\, a \in A,\, b \in B\}$. Lets consider an alternative definition of supremum that you have seen in section. If $\gamma = \sup C$ then for all $\varepsilon > 0$ $\exists c \in C : \gamma - \varepsilon < c$. So, lets fix $\varepsilon > 0$ and show how to find such $c \in C$.

   Lets construct a sequence $\{a_n\}$ and $\{b_n\}$ such that for all $n \in \mathbf{N} : a_n \in A$ and $b_n \in B$. Moreover, by the alternative definition of supremum we can pick such a_n and b_n that $\alpha - 1/n < a_n$ and $\beta - 1/n < b_n$ (of course, a sequence can be trivial after some $n \in \mathbf{N}$). By construction $a_n \to \alpha$ and $b_n \to \beta$ as $n \to \infty$ and, therefore, $a_n \cdot b_n \to \alpha \cdot \beta$ by property of limits. Also, $a_n \cdot b_n \in C$ for all $n \in \mathbf{N}$. Now note that product is a continuous function, therefore, given $\varepsilon > 0$ we can find such a'_n and b'_n in a sufficiently small neighborhood of $(\alpha, \beta) \in \mathbf{R}^{\mathbf{2}}_{+}$ such that $|\alpha \cdot \beta - a'_n \cdot b'_n| < \varepsilon$. But that is exactly what we need to show since our $c = a'_n \cdot b'_n$ for those sufficiently large n.

7. Let X be any nonempty set. If a distance function $d : X \times X \to \mathbf{R}_+$ that satisfies assumptions of *(i)* symmetry, *(ii)* triangle inequality and *(iii)* $d(x, x) = 0$ for all $x \in X$, then we say that d is a *semi-metric* on X, and (X, d) is a *semi-metric* space.
For any semi-metric space X, define the binary relation $\approx$ on X by $x \approx y$ iff $d(x, y) = 0$. Now, define $[x] = \{ y \in Y : x \approx y\}$ for all $x \in X$, and let $\mathcal{X} = \{[x] : x \in X\}$. Finally, define $D : \mathcal{X} \times \mathcal{X} \to \mathbf{R}_+$ by $D([x], [y]) = d(x, y)$.

(a) Show that $\approx$ is an equivalence relation on X.

Solution.

Reflexivity and symmetry are immediate given the definition of a semi-metric (assumptions (iii) and (i)).

Transitivity. We need to show that for all $x, y, z \in X$, $(x \approx y \wedge y \approx z) \implies x \approx z$. Lets suppose we have $x, y, z \in X$ such that above condition holds. $x \approx y \iff d(x, y) = 0$ and $y \approx z \iff d(y, z) = 0$ immediately implies $x \approx z$ as we need, by the triangle inequality for the norm $d(x, z) \le d(x, y) + d(y, z)$.

(b) Prove that $(\mathcal{X}, D)$ is a metric space.

Solution. First, lets show that the distance function $D(\cdot, \cdot)$ is well-defined. Lets pick $x \in [x]$, $y \in [y]$ and $x' \in [x]$, $y' \in [y]$, such that $x \neq x'$ and $y \neq y'$. We would like to show that $D([x], [y]) = D([x'], [y'])$, or, by definition of $D(\cdot, \cdot)$, that $d(x, y) = d(x', y')$. Applying triangle inequality twice we get

$$d(x, y) \le d(x, x') + d(x', y) \le d(x, x') + d(x', y') + d(y', y)$$
$$d(x', y') \le d(x', x) + d(x, y') \le d(x', x) + d(x, y) + d(y, y')$$

Noting that $d(x, x') = d(y, y')$ because both $x, x' \in [x]$ and $y, y' \in [y]$ we get $d(x, y) \le d(x', y')$ and $d(x, y) \ge d(x', y')$. Thus, $d(x, y) = d(x', y')$, and $D(\cdot, \cdot)$ is, indeed, well-defined. Now, lets check whether $D(\cdot, \cdot)$ satisfies all properties of the distance function.

$D([x], [y]) = 0 \iff [x] = [y]$. Suppose $D([x], [y]) = 0$. By definition this implies $d(x, y) = 0$ iff $x \approx y$, or $[x] = [y]$.

Symmetry. Follows directly because any semi-norm is symmetric.

Triangle Inequality. We need to show that for any three equivalence classes $[x], [y], [z]$ we have $D([x], [z]) \le D([x], [y]) + D([y], [z])$. So, fix some x, y and z in X, such that $[x] \neq [y] \neq [z]$. It is easy to see that triangle inequality for $D(\cdot, \cdot)$ follows immediately from triangle inequality for $d(\cdot, \cdot)$ since $d(x, z) \le d(x, y) + d(y, z)$.

8. Let c_{00} be the space of all finite sequences.

   (a) Show that for $p \in [1, +\infty)$ the real-valued operation $\|\cdot\|_p$

$$\|x\|_p = \left(\sum_{n=0}^{\infty} |x_n|^p \right)^{\frac{1}{p}}$$

is a norm.

Solution. Note that given how $\|\cdot\|_p$ is defined, all properties of a norm are immediate except for triangle inequality. For the latter, we prove the Minkowski's inequality, i.e. that for any x_m, $y_m \in \mathbf{R}^\infty$ and $1 \leq p < \infty$

$$\left(\sum_{i=1}^{\infty} |x_i + y_i|^p \right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^{\infty} |y_i|^p \right)^{\frac{1}{p}}.$$

Take and x_m, $y_m \in \mathbf{R}^\infty$ and fix any $1 \leq p < \infty$. If either $\sum^\infty |x_i|^p = \infty$ or $\sum^\infty |y_i|^p = \infty$, then Minkowski's inequality is trivial, so we assume it is not the case and $(\sum^\infty |x_i|^p)^{1/p}$ and $(\sum^\infty |y_i|^p)^{1/p}$ are some positive real numbers, say α and β, respectively.
Define the real sequences x'_m and y'_m by $x'_m = \frac{1}{\alpha}|\,x_m|$ and $y'_m = \frac{1}{\beta}|\,y_m|$. Notice that $\sum^\infty |x'_i|^p = \sum^\infty |y'_i|^p = 1$. Using the triangle inequality for the absolute value function and the fact that $t \to t^p$ is an increasing map on $\mathbf{R}_+$ we get

$$|\,x_i + y_i|^p \leq (|\,x_i| + |\,y_i|)^p = (\alpha|\,x'_i| + \beta|\,y'_i|)^p = (\alpha+\beta)^p \left(\frac{\alpha}{\alpha+\beta}|\,x'_i| + \frac{\alpha}{\alpha+\beta}|\,y'_i| \right)^p.$$

for each $i = 1, 2, \ldots$.
Since $t \to t^p$ is a convex map on $\mathbf{R}_+$, we have

$$\left(\frac{\alpha}{\alpha+\beta}|\,x'_i| + \frac{\beta}{\alpha+\beta}|\,y'_i| \right)^p \leq \frac{\alpha}{\alpha+\beta}|\,x'_i|^p + \frac{\beta}{\alpha+\beta}|\,y'_i|^p$$

and hence

$$|\,x_i + y_i|^p \leq (\alpha+\beta)^p \left(\frac{\alpha}{\alpha+\beta}|\,x'_i|^p + \frac{\beta}{\alpha+\beta}|\,y'_i|^p \right).$$

Summing over i we get

$$\begin{aligned} \sum_{i=1}^{\infty} |\,x_i + y_i|^p &\leq (\alpha+\beta)^p \left(\frac{\alpha}{\alpha+\beta} \sum_{i=1}^{\infty} |\,x'_i|^p + \frac{\beta}{\alpha+\beta} \sum_{i=1}^{\infty} |\,y'_i|^p \right) \\ &= (\alpha+\beta)^p \left(\frac{\alpha}{\alpha+\beta} + \frac{\beta}{\alpha+\beta} \right). \end{aligned}$$

Thus, $\sum_{i=1}^{\infty} |\,x_i + y_i|^p \leq (\alpha+\beta)^p$ and we get the results we desire.

(b) Are $||\cdot||_p$ equivalent to $||\cdot||_q$ on c_{00} for $p \neq q$?

Solution. No they are not. In a few days you will learn about a basis of a vector space and its dimensionality, but for now, lets just note that c_{00} is an infinite dimensional space. Because of that, it supports norms that are not Lipschitz equivalent and this exercise is just an example for that. (What would be a basis of c_{00}?)

Lets fix p and q such that $1 \leq p < q < \infty$. Lets consider a sequence:

$$x_k^{(n)} = \begin{cases} \left(\frac{1}{k}\right)^{\frac{1}{p}}, & k \leq n \\ 0, & k > n \end{cases}$$

Note that such $\{x^{(n)}\} \in c_{00}$ for any finite $n \in \mathbf{N}$. Also, $\left(\frac{1}{n^{1/p}}\right)^p = \frac{1}{n}$ and $\left(\frac{1}{n^{1/p}}\right)^q = \frac{1}{n^\alpha}$ where a power of $\alpha > 1$. Therefore, $\lim_{n\to\infty} ||\, x^{(n)}||_p = \infty$ and $\lim_{n\to\infty} ||\, x^{(n)}||_q < \infty$ because of our assumption on p and q. Thus, we can never find an m, $M > 0$ such that

$$m \leq \frac{||\, x^{(n)}||_p}{||\, x^{(n)}||_q} \leq M$$

as the ratio of the norms is unbounded.

Finally, observe that in our case, non-equivalence of norms gives us a convergence with respect to one norm, but not the other. When norms are Lipschitz equivalent, they must all generate the same type of convergence in vector space, in other words, equivalent norms give rise the same *topological* properties (because we can describe convergence properties equivalently in terms of open neighborhoods.)