

**Problem 1.**

Let  $A, B, C$  be sets. Prove the following statements:

$$(a) \ C \setminus (A \cup B) = (C \setminus A) \cap (C \setminus B)$$

$$(b) \ C \setminus (A \cap B) = (C \setminus A) \cup (C \setminus B)$$

**Solution**

To prove equality of sets you must show that the left-hand side (LHS) is contained in the right-hand side (RHS) and vice-versa:

(a) LHS  $\subseteq$  RHS: Let  $x \in C \setminus (A \cup B)$ . Then  $x \in C$  and  $x \notin (A \cup B)$ . If  $x \in A$  or  $x \in B$  then  $x \in (A \cup B)$ ; hence we must have  $x \notin A$  and  $x \notin B$  (De Morgan's Law). Then  $x \in C$  and  $x \notin A$  implies  $x \in C \setminus A$ . Similarly  $x \in C$  and  $x \notin B$  implies  $x \in C \setminus B$  as well. So  $x \in (C \setminus A) \cap (C \setminus B)$ .

RHS  $\subseteq$  LHS: Let  $x \in (C \setminus A) \cap (C \setminus B)$ . Then  $x \in C$ ,  $x \notin A$  and  $x \notin B$ . If  $x \in A \cup B$  then either  $x \in A$  or  $x \in B$ ; hence we must have  $x \notin (A \cup B)$  (De Morgan's Law again). But  $x \in C$  and  $x \notin (A \cup B)$  says that  $x \in C \setminus (A \cup B)$ .

(b) LHS  $\subseteq$  RHS: Let  $x \in C \setminus (A \cap B)$ . Then  $x \in C$  and  $x \notin (A \cap B)$ . If  $x \in A$  and  $x \in B$  then  $x \in (A \cap B)$ ; hence we must have  $x \notin A$  or  $x \notin B$ . Then we have either  $x \in C$  and  $x \notin A$ , or we have  $x \in C$  and  $x \notin B$ . So  $x \in (C \setminus A) \cup (C \setminus B)$ .

RHS  $\subseteq$  LHS: Let  $x \in (C \setminus A) \cup (C \setminus B)$ . Then either  $x \in C$  and  $x \notin A$ , or  $x \in C$  and  $x \notin B$ . If  $x \in (A \cap B)$  then  $x \in A$  and  $x \in B$ ; hence we must have  $x \notin (A \cap B)$ . But  $x \in C$  and  $x \notin (A \cap B)$  says that  $x \in C \setminus (A \cap B)$ .

**Remark:** It's ok to for example show part (b) as follows:

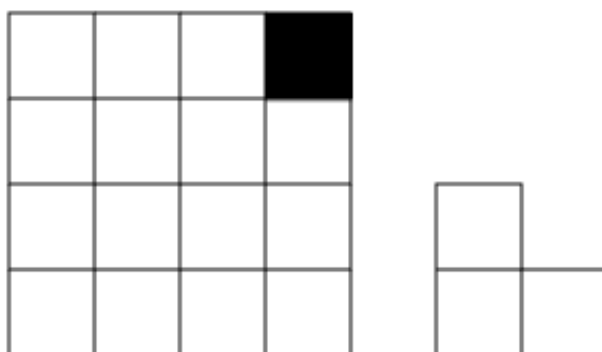
$$\begin{aligned} C \setminus (A \cap B) &= C \cap (A \cap B)^c \\ &= C \cap (A^c \cup B^c) \\ &= (C \cap A^c) \cup (C \cap B^c) \\ &= (C \setminus A) \cup (C \setminus B) \end{aligned}$$

But be sure you could prove any of those particular steps (first and last equalities are definitions; the second is De Morgan's law and the third uses the fact that intersection distributes over union).

## Problem 2.

Use the principle of mathematical induction to prove the following statements:

- (a) A set  $S$  with  $n$  elements has  $2^n$  subsets. (note: do not forget about the empty set)
- (b)  $\left| \sum_{n=1}^N x_n \right| \leq \sum_{n=1}^N |x_n|$ , for  $x_n \in \mathbb{R}$
- (c) Prove that any grid made up of  $2^n \times 2^n$  tiles can be covered except for one corner tile by L-shaped triominoes (the triominoes may be rotated). The figure below shows an example of a  $4 \times 4$  grid (left) where all of the non-shaded tiles must be covered by a triomino (right). Note: Visual proofs of the base and inductive steps are fine.



## Solution

- (a) **Base step** ( $n = 0$ ): The set containing 0 elements is the empty set. Since the only subset of the empty set is itself, we have  $\mathcal{P}(\emptyset) = \{\emptyset\}$ . Hence  $|\mathcal{P}(S)| = 1 = 2^0$ . Thus the claim holds for  $n = 0$ .

**Inductive hypothesis** ( $n = k$ ): If  $|S| = k$ , then  $|\mathcal{P}(S)| = 2^k$ .

**Inductive step:** Take any set  $S$  such that  $|S| = k + 1$ . Fix some element of  $S$  (call it  $s_{k+1}$ ) and consider the power set of  $S \setminus \{s_{k+1}\}$ , which I denote  $X_1 = \mathcal{P}(S \setminus \{s_{k+1}\})$ . Note that  $A \subset S \setminus \{s_{k+1}\}$  if and only if  $A \subset S$  and  $s_{k+1} \notin A$ . In other words,  $X_1 = \{A : A \subset S \text{ and } s_{k+1} \notin A\}$ , so  $X_1 \subsetneq \mathcal{P}(S)$ . By our induction hypothesis,  $|X_1| = 2^k$ .

Now consider the (nonempty) set  $X_2 = \mathcal{P}(S) \setminus X_1$ . We have  $A \subset S$  and  $A \notin X_1$  if and only if  $A \subset S$  and  $s_{k+1} \in A$ , so  $X_2 = \{A : A \subset S \text{ and } s_{k+1} \in A\}$ . Define the following function  $f : X_1 \rightarrow X_2$  as follows: for any  $A \in X_1$ ,  $f(A) = A \cup \{s_{k+1}\}$ .  $f$  is one-to-one, since if  $A \neq B$  and  $s_{k+1}$  is not an element of either  $A$  or  $B$ , then  $A \cup \{s_{k+1}\} \neq B \cup \{s_{k+1}\}$ .  $f$  is also onto, since every element of  $X_2$  is of the form  $A \cup \{s_{k+1}\}$  for some  $A \in X_1$  (including the empty set!). Hence we have defined a bijection from  $X_1$  to  $X_2$ , so  $|X_1| = |X_2|$ . Then since  $X_1$  and  $X_2$  are finite and partition  $\mathcal{P}(S)$ , we have  $|\mathcal{P}(S)| = |X_1| + |X_2| = 2^k + 2^k = 2^{k+1}$ .

- (b) **Base step** ( $n = 2$ ): This just says  $|x_1 + x_2| \leq |x_1| + |x_2|$ , which is the triangle inequality. One proof:

$$\begin{aligned} |x_1 + x_2|^2 &= (x_1 + x_2)^2 \\ &= x^2 + 2xy + y^2 \\ &\leq |x|^2 + 2|x||y| + |y|^2 \\ &= (|x| + |y|)^2 \end{aligned}$$

Since squaring preserves order (try to use the properties you proved in problem 5 to show this), we have  $|x_1 + x_2| \leq |x_1| + |x_2|$ .

**Inductive hypothesis** ( $n = k$ ):  $\left| \sum_i^k x_i \right| \leq \sum_i^k |x_i|$ .

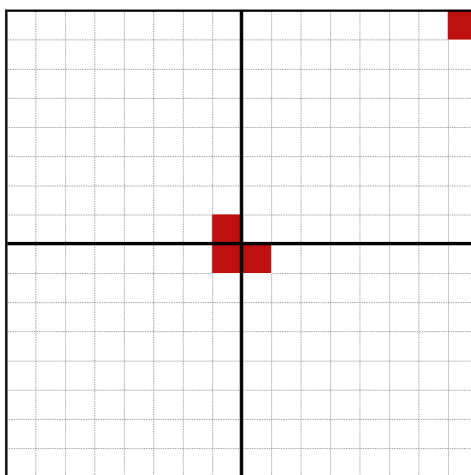
**Inductive step:** The triangle inequality gives us that  $\left| \sum_i^{k+1} x_i \right| = \left| \sum_i^k x_i + x_{k+1} \right| \leq \left| \sum_i^k x_i \right| + |x_{k+1}|$ . So then from the inductive hypothesis we have  $\left| \sum_i^k x_i \right| + |x_{k+1}| \leq \sum_i^k |x_i| + |x_{k+1}| = \sum_i^{k+1} |x_i|$ .

**Remark:** Note that the statement holds trivially for  $n = 1$ , but the logic of the inductive step only applies to  $n \geq 2$ , so we need to prove directly the case of  $n = 2$ .

- (c) **Base step** ( $n = 1$ ): A single triomino placed on a  $2 \times 2$  grid works.

**Inductive hypothesis** ( $n = k$ ): Any  $2^k \times 2^k$  grid can be covered by triominoes, except for one corner square.

**Inductive step:** Note that a  $2^{k+1} \times 2^{k+1}$  is really just four different  $2^k \times 2^k$  grids. By the inductive hypothesis, cover each of these grids by triominoes (except for a corner square) and organize the blocks such that three of the uncovered corner squares are in the center of the grid, while the fourth is on the corner of the larger  $2^k \times 2^k$  grid. Then we can cover the three center squares by one more triomino. See figure:



### Problem 3.

Let  $A$  and  $B$  be subsets of any uncountable set  $X$  such that their complements are countably infinite. Prove that  $A \cap B \neq \emptyset$ .

#### Solution

Remember a set  $S$  is countably infinite if there exists a bijection  $f : \mathbb{N} \rightarrow S$ ; and if  $S$  is either finite or countably infinite, then  $S$  is countable. Otherwise we say  $S$  is uncountable.

Toward contradiction, assume that  $A \cap B = \emptyset$ . This is equivalent to saying  $A \subseteq B^c$ . But since  $A$  is a subset of a countably infinite set,  $A$  is also countable. To see why: first if  $A$  is finite then we're done. If instead  $A$  is infinite, then since  $B^c$  is countably infinite choose some bijection  $f : \mathbb{N} \rightarrow B^c$ . Then define the following function  $g : \mathbb{N} \rightarrow A$ :  $g(1) = f(n_1)$  where  $n_1$  is the smallest natural number such that  $f(n_1) \in A$ . Then given  $n_1, \dots, n_{k-1}$ , let  $n_k$  be the smallest natural number greater than  $n_{k-1}$  such that  $f(n_k) \in A$ . Let  $g(k) = f(n_k)$ . This defines a bijection from  $\mathbb{N}$  into  $A$ .

Now I'll show that  $A \cup A^c$  is also countably infinite. If  $A$  is finite this is easy. So instead suppose  $A$  is also countably infinite. So we can find bijections  $f : \mathbb{N} \rightarrow A$  and  $g : \mathbb{N} \rightarrow A^c$ . Consider the following function  $h : \mathbb{N} \rightarrow A \cup A^c$ , where  $h(n) = f(\frac{n}{2})$  if  $n$  is even and  $h(n) = g(\frac{n+1}{2})$  if  $n$  is odd. Then  $h$  is one-to-one: if  $m \neq n$  then  $f(m) \neq f(n)$ ,  $g(m) \neq g(n)$  (since  $f, g$  are bijections), and  $f(m) \neq g(n)$  (since  $f, g$  have disjoint codomains), hence  $h(m) \neq h(n)$ . And  $h$  is onto: for any  $x \in A \cup A^c$  there exists an  $n$  such that either  $f(\frac{n}{2}) = x$  or  $g(\frac{n+1}{2}) = x$ , so  $h(n) = x$ . Thus  $A \cup A^c$  is countable.

But  $A \cup A^c = X$  and  $X$  is uncountable, a contradiction. Thus  $A \cap B \neq \emptyset$ .

**Remark:** There was nothing special about the sets  $A$  and  $B^c$ , so we have shown that for any countable set  $T$ , if  $S \subset T$  then  $T$  is countable. Also we showed that the union of two countable and disjoint sets is also countable; think about how you could adapt the proof to show that the union of any two countable sets is countable. Then try to extend this to the union of any finite collection of countable sets, or even to any countable union of countable sets.

**Problem 4.**

If  $x \neq 0$  is rational and  $y$  is irrational, prove that  $x + y$  and  $x \cdot y$  are irrational. If  $x \neq 0$  is instead irrational, does the statement still hold?

**Solution**

Since  $\mathbb{Q}$  is a field, we have that the sum, product, additive inverse and multiplicative inverse of any rational numbers are also rational.

Then towards contradiction, suppose  $x + y$  is rational. Since  $x$  is rational,  $-x$  is rational, and the sum  $-x + (x + y) = (-x + x) + y = 0 + y = y$  is also rational, a contradiction.

Now suppose  $x \cdot y$  is rational. Since  $x$  is rational,  $x^{-1}$  is rational, and the product  $x^{-1} \cdot (x \cdot y) = (x^{-1} \cdot x) \cdot y = 1 \cdot y = y$  is rational, a contradiction.

However the sums and products of irrational numbers certainly may be rational. A consequence of the uniqueness of additive and multiplicative inverses is that if  $y$  is irrational, then  $-y$  and  $y^{-1}$  are also irrational (if not then  $-(-y)$  and  $(y^{-1})^{-1}$  would be rational!). But  $-y + y = 0 \in \mathbb{Q}$  and  $y^{-1} \cdot y = 1 \in \mathbb{Q}$ .

**Remark:** In the proof above I took as given that  $\mathbb{Q}$  is a field, you should be able to prove this is so. As an example, let's show that for any  $r \in \mathbb{Q}$  we have  $r^{-1} \in \mathbb{Q}$ . Remember a rational number can be written as the ratio of two integers, or more explicitly as the product of an integer and the multiplicative inverse of an integer. So let  $r = m \cdot n^{-1}$ . Then

$$\begin{aligned} r + -(m \cdot n^{-1}) &= m \cdot n^{-1} + -(m \cdot n^{-1}) \\ &= m \cdot n^{-1} + ((-m) \cdot n^{-1}) \\ &= (m - m) \cdot n^{-1} = 0 \end{aligned}$$

So by the uniqueness of additive inverses, we have  $-r = -(m \cdot n^{-1})$ .

**Problem 5.**

Recall the definition of an *ordered field*: a field  $F$  with a binary relation “ $\leq$ ” such that  $\forall x, y, z \in F$ , we have:

- Totality:  $x \leq y$  or  $y \leq x$
- Antisymmetry:  $x \leq y$  and  $y \leq x \implies x = y$
- Transitivity:  $x \leq y$  and  $y \leq z \implies x \leq z$
- The order complies with addition and multiplication:  $y \leq z \implies x + y \leq x + z$  and  $x \geq 0, y \geq 0 \implies x \cdot y \geq 0$

We define “ $x < y$ ” as “ $x \leq y$ ” but not “ $y \leq x$ ”; similarly for  $x > y$ .

(a) Prove the following properties of any ordered field:

- (i)  $x \geq 0 \implies -x \leq 0$  and vice versa.
- (ii)  $x \geq 0$  and  $y \leq z \implies x \cdot y \leq x \cdot z$
- (iii)  $x \leq 0$  and  $y \leq z \implies x \cdot y \geq x \cdot z$
- (iv)  $x \neq 0 \implies x^2 > 0$
- (v)  $0 < x < y \implies 0 < y^{-1} < x^{-1}$

(b) Using the above properties, prove that the complex field  $\mathbb{C}$  cannot be made into an ordered field.

**Solution**

(a) In section we showed  $(-x)y = -(xy) = x(-y)$  (call this property  $\star$ ). Further,  $\star$  implies that  $(-x)(-y) = -(x(-y)) = -(-(xy)) = xy$ , where the final equality follows from the uniqueness of the additive inverse (call this property  $\star\star$ ).

- (i) If  $x \geq 0$  then  $0 = -x + x \geq -x + 0 = -x$ . The other direction is the same.
- (ii) If  $z \geq y$  then  $z - y \geq y - y = 0$ . Then since  $x \geq 0$ , we have  $x(z - y) \geq 0$ . Then
 
$$xz = xz + (-xy + xy) = (xz - xy) + xy = x(z - y) + xy \geq 0 + xy = xy \quad (1)$$
- (iii) Since  $x \leq 0$ , (i) implies  $-x \geq 0$ . Then from  $\star$ ,  $-(x(z - y)) = -x(z - y) \geq 0$ . But then (i) implies  $x(z - y) \leq 0$ . The same argument as in (1) now shows  $xz \leq xy$ .
- (iv) If  $x > 0$  then  $x^2 > 0$ . If  $x < 0$  then  $-x > 0$  so  $(-x)^2 > 0$ .  $\star\star$  gives that  $(-x)^2 = x^2$ .
- (v) Note that (iv) implies  $1 > 0$ . Since  $y > 0$  and  $y^{-1}y = 1$ , we have  $y^{-1} > 0$  (to see why, suppose  $y^{-1} \leq 0$ ; then (iii) says  $0 \leq 1 \implies 0 = 0 \cdot y^{-1} \geq y^{-1}y = 1$ ). Similarly  $x^{-1} > 0$ , so  $x^{-1}y^{-1} > 0$ . Then  $x < y \implies (x^{-1}y^{-1})x < (x^{-1}y^{-1})y \implies y^{-1} < x^{-1}$ .

(b) The complex number  $i$  satisfies  $i^2 = -1$ . Since  $1 > 0$ ,  $-1 < 0$ . Hence any potential order we could define over the complex field would fail property (iv).

**Problem 6.**

Let  $A$  be a subset of  $\mathbb{R}$  that is nonempty and bounded below. Define the set  $-A = \{-a : a \in A\}$ . Prove that  $\inf A = -\sup(-A)$ .

**Solution**

Let  $\beta = \inf A$ . So  $\beta \leq a$  for every  $a \in A$ . From problem 5 we have that  $-\beta \geq -a$  for every  $a \in A$ , so  $-\beta$  is an upper bound of  $-A$ . Now we show that any upper bound of  $-A$  is greater than  $-\beta$ . Choose some upper bound  $u$  of  $-A$ . That is to say, for every  $a \in A$  we have  $u \geq -a$ . But then  $-u \leq a$  so  $-u$  is a lower bound of  $A$ . Hence  $-u \leq \beta \implies u \geq -\beta$ . Since  $u$  was an arbitrary upper bound, we have that  $-\beta = \sup -A$ .

**Problem 7.**

Define the following distance function on the set of real numbers:

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

- (a) Prove that  $(\mathbb{R}, d)$  is a metric space.
- (b) Identify the open (and closed) balls in the topology induced by this metric.

**Solution**

- (a) To verify that  $d$  is a metric, you need to check that for all  $x, y, z \in \mathbb{R}$ , (i)  $d(x, x) = 0$ , (ii)  $d(x, y) = d(y, x)$ , and (iii)  $d(x, y) + d(y, z) \geq d(x, z)$ . Requirements (i) and (ii) are easily verified. To verify (iii) there are two cases to consider:  $x = z$  or  $x \neq z$ .

**Case I:** If  $x \neq z$ , then either  $x \neq y$  or  $y \neq z$  (why?  $x = y$  and  $y = z$  implies  $x = z$ , so take the contrapositive). Thus either  $d(x, y) = 1$  or  $d(y, z) = 1$  and we have  $d(x, y) + d(y, z) \geq 1 = d(x, z)$ .

**Case II:** If  $x = z$  then  $d(x, z) = 0 \leq d(x, y) + d(y, z)$ .

So (iii) holds and  $d$  is a metric.

- (b) Given any point  $x$  consider the ball centered at  $x$  with radius  $\varepsilon$ . When  $\varepsilon \leq 1$  we have  $B_\varepsilon(x) = \{y \in \mathbb{R} : d(x, y) < \varepsilon\} = \{x\}$  since  $d(x, y) < \varepsilon \leq 1 \implies d(x, y) = 0 \implies x = y$ . Similarly, when  $\varepsilon > 1$  we obtain  $B_\varepsilon(x) = \{y \in \mathbb{R} : d(x, y) < \varepsilon\} = \mathbb{R}$  since  $\forall y \in \mathbb{R}$  where  $y \neq x$ , then  $d(x, y) = 1 < \varepsilon$ . Therefore, the open balls in this space either look like singleton points  $\{x\}$  or the entire space. For  $\varepsilon < 1$ , the closed ball coincides with the open ball of radius  $\varepsilon$ . For  $\varepsilon \geq 1$ , the closed ball is the entire space.

**Remark:** Note that the analysis above shows that points are *open* in this space. Since arbitrary unions of open sets are open it follows that every subset of this space is open. On the other hand, since a closed set is defined to be the complement of an open set, every subset of this space is also closed. A space with such a topology (where every point is both open and closed) is said to be *discrete*.