Economics 204
Fall 2012
Problem Set 2
Due Tuesday, July 31 in Lecture

1. Let (Y, d) be a metric space. Call x a *limit* point of some set $X \subseteq Y$ if every open ball around x contains another element of X distinct from x.[1] Suppose that $X \subseteq \mathbf{R}$ be uncountable. Without invoking any compactness arguments, prove that X has at least one limit point.[2]

2. Let (X, d) be a metric space, where $X \subseteq \mathbf{R}$ and d is a standard Euclidean metric. Give an example of a non-trivial set in X which is both open and closed.

3. Identify the set of interior points, limit points, isolated points, and boundary points of the following sets. Assume the metric is Euclidean unless indicated otherwise (no proofs necessary):

   (a) $\{1, 1/2, 1/3, 1/4, ...\} \cup \{-1, -1/2, -1/3, -1/4, ...\} \cup \{0\} \subset \mathbb{R}$ (i.e. the ambient space is $\mathbb{R}$)

   (b) $\mathbb{N} \subset \mathbb{R}$

   (c) $\mathbb{N} \subset \mathbb{R}$ with discrete metric[3]

   (d) $\mathbb{Q} \subset \mathbb{R}$

   (e) $\mathbb{Q} \subset \mathbb{R}$ with discrete metric

   (f) $\{x \in \mathbb{Q} : x < \pi\} \subset \mathbb{R}$

   (g) $\{x \in \mathbb{Q} : x < \pi\} \subset \mathbb{Q}$

4. Show that any closed set in a metric space is an intersection of a decreasing sequence of open sets. Show that any open set is a union of an increasing sequence of closed sets.

[1] Notice that this is more restrictive than the definition of closure point, because now the intersection of X and $B_\epsilon(x)$ cannot be just the point x itself. Points for which this is the case are called *isolated* points. Hence, the union of those two sets, limit points and isolated points, is the closure of the set X. You can read more about that in de la Fuente p. 41.

[2] You will have a chance to use compactness in showing this fact on problem set 3.

[3] Recall that we defined discrete metric as $d(x, y) = 0$ iff $x = y$ and $d(x, y) = 1$ if $x \neq y$.

5. Give an example of function $f : \mathbf{R} \to \mathbf{R}$ which is

   (a) *nowhere* continuous (i.e. discontinuous for all $x \in \mathbf{R}$), but the absolute value of which is, in fact, continuous. Please use standard Euclidean metric, there is no need to be excessively creative.

   (b) continuous at exactly one point? two points? n points?

6. Consider a real-valued continuous function f defined on interval $[a, b]$ with a property that $f(a) = a$ and $f(b) = b$. Let g be any continuous function that maps $[a, b]$ into itself. Prove that there is $x^* \in [a, b]$ such that $f(x^*) = g(x^*)$. Will the statement hold if we just assume that g is a continuous function on $[a, b]$, but not necessarily maps $[a, b]$ into itself? Prove or give counter-example.

7. Suppose that $\{f_n\}$ is a sequence of non-decreasing functions that map the unit interval into itself. Suppose that

$$\lim_{n \to +\infty} f_n(x) = f(x)$$

   pointwise and f is a continuous function. Prove that the convergence of $f_n(x)$ to $f(x)$ is uniform, i.e. prove that[4]

$$\forall \epsilon > 0 \; \exists N_\epsilon : n > N_\epsilon \quad |f_n(x) - f(x)| < \epsilon \text{ for all } x \in [0, 1].$$

8. Let (X, d) be a metric space. Let $\{x_n\}$ and $\{y_n\}$ be two Cauchy sequences in X. Call $\{x_n\}$ and $\{y_n\}$ *Cauchy equivalent* if $x_0, y_0, x_1, y_1, \ldots$ is a Cauchy sequence itself.

   (a) Prove that $\{x_n\}$ and $\{y_n\}$ are Cauchy equivalent iff $\lim_{n\to\infty} d(x_n, y_n) = 0$.

   (b) Let $\{x_n\}$ and $\{y_n\}$ be two Cauchy equivalent sequences and $\{z_n\}$ another Cauchy sequence. Prove that

$$\lim_{n\to\infty} d(x_n, z_n) = \lim_{n\to\infty} d(y_n, z_n).$$

   (c) Show that equivalence of Cauchy sequences is an equivalence relation on X.

   (d) Let X^* be a set of equivalence classes of Cauchy sequences in X. Prove that the function

$$\{x_n\}, \{y_n\} \to \lim_{n\to\infty} d(x_n, y_n)$$

   defines a metric on X^*.

[4] Like with uniform continuity, same ϵ works for all x in uniform convergence, whereas in pointwise convergence, ϵ will, in general, depend on the choice of x.