Economics 204
Fall 2012
Problem Set 2 Suggested Solutions

1. Let (Y, d) be a metric space. Call x a *limit* point of some set $X \subseteq Y$ if every open ball around x contains another element of X distinct from x.[1] Suppose that $X \subseteq \mathbf{R}$ be uncountable. Without invoking any compactness arguments, prove that X has at least one limit point.[2]

   Solution. Let $k \in \mathbf{Z}$ and consider sets of the form $C_k = X \cap [k, k+1)$. Clearly, $X = \cup_{k \in \mathbf{Z}} C_k$ and because X is uncountable, at least one C_k must contain an uncountably many elements of X (if not, then X is a countable union of at most countable sets, thus, countable.) Observe that each C_k is bounded and by Bolzano-Weierstrass theorem there is a convergent subsequence in those sets C_k that contain uncountably many elements. The limit of that subsequence is a limit point of X.

   Now, there are is guarantee that the subsequence that we get is not the same element repeated infinitely. To remedy this, we pick our initial sequence such that all its elements are distinct from each other.

2. Let (X, d) be a metric space, where $X \subseteq \mathbf{R}$ and d is a standard Euclidean metric. Give an example of a non-trivial set in X which is both open and closed.

   Solution. $X = [0, 1] \cup [2, 3]$. Each of the closed intervals is both open and closed in X.

3. Identify the set of interior points, limit points, isolated points, and boundary points of the following sets. Assume the metric is Euclidean unless indicated otherwise (no proofs necessary):

   Solution. For each set X, answers are provided in the following format:

   $$\{\,\text{Int}X,\, \{\text{limit points of } X\},\, \{\,\text{isolated points of } X\},\, \partial X\}$$

   For the sake of brevity, in a few places X is used the denote the entire set in question.

   (a) $\{1, 1/2, 1/3, 1/4, ...\} \cup \{-1, -1/2, -1/3, -1/4, ...\} \cup \{0\} \subset \mathbb{R}$ (i.e. the ambient space is $\mathbb{R}$)

   Solution. $\{\emptyset, \{0\}, X \setminus \{0\}, X\}$

[1] Notice that this is more restrictive than the definition of closure point, because now the intersection of X and $B_\epsilon(x)$ cannot be just the point x itself. Points for which this is the case are called *isolated* points. Hence, the union of those two sets, limit points and isolated points, is the closure of the set X. You can read more about that in de la Fuente p. 41.

[2] You will have a chance to use compactness in showing this fact on problem set 3.

(b) $\mathbb{N} \subset \mathbb{R}$

Solution. $\{\emptyset,\, \emptyset,\, X,\, X\}$

(c) $\mathbb{N} \subset \mathbb{R}$ with discrete metric[3]

Solution. $\{X,\, \emptyset,\, X,\, \emptyset\}$

(d) $\mathbb{Q} \subset \mathbb{R}$

Solution. $\{\emptyset,\, \mathbb{R},\, \emptyset,\, \mathbb{R}\}$

(e) $\mathbb{Q} \subset \mathbb{R}$ with discrete metric

Solution. $\{X,\, \emptyset,\, X,\, \emptyset\}$

(f) $\{x \in \mathbb{Q} : x < \pi\} \subset \mathbb{R}$

Solution. $\{\emptyset,\, \{x \in \mathbb{R} : x \leq \pi\},\, \emptyset,\, \{x \in \mathbb{R} : x \leq \pi\}\}$

(g) $\{x \in \mathbb{Q} : x < \pi\} \subset \mathbb{Q}$

Solution. $\{X,\, X,\, \emptyset,\, \emptyset\}$

4. Show that any closed set in a metric space is an intersection of a decreasing sequence of open sets. Show that any open set is union of an increasing sequence of closed sets.

Solution. Let C be closed in a metric space (X, d). We need to show that there is a sequence of open sets $\{U_n\}_{n=1}^{\infty}$ such that

$$U_1 \supseteq U_2 \supseteq U_3 \supseteq \ldots$$

and $C = \cap_n U_n$.
Consider $U_n = \cup_{x \in C} B_{\frac{1}{n}}(x)$. Obviously, any such U_n is open as a union of open sets and by construction $\{U_n\}_{n=1}^{\infty}$ is a decreasing sequence. Lets show that $C = \cap_n U_n$ by demonstrating two way set inclusion.
($\subseteq$) Take any $x \in C$. By definition $x \in U_n$ for all n, thus, $x \in \cap_n U_n$.
($\supseteq$) Now, take $x \in \cap_n U_n$ and suppose that $x \notin C$. Then $x \in X \setminus C^c$ and because $X \setminus C^c$ is open, there is an $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq X \setminus C^c$. But this implies that for all $k > 2/\varepsilon$ $x \notin U_k \implies x \notin \cap_n U_n$.

Second part of the statement follows immediately by considering complements.

[3]Recall that we defined discrete metric as $d(x, y) = 0$ iff $x = y$ and $d(x, y) = 1$ if $x \neq y$.

5. Give an example of function $f : \mathbf{R} \to \mathbf{R}$ which is

   (a) *nowhere* continuous (i.e. discontinuous for all $x \in \mathbf{R}$), but the absolute value of which is, in fact, continuous.[4]

   Solution.
$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbf{Q} \\ -1, & \text{if } x \notin \mathbf{Q}. \end{cases}$$

   (b) continuous at exactly one point? two points? n points?

   Solution. Continuous at exactly one point
$$f(x) = \begin{cases} x, & \text{if } x \in \mathbf{Q} \\ -x, & \text{if } x \notin \mathbf{Q}. \end{cases},$$

   For $n > 1$ points of continuity consider
$$f(x) = \begin{cases} \sin x, & \text{if } x \in \mathbf{Q} \text{ and } x < \pi(n-1), \\ -\sin x, & \text{if } x \notin \mathbf{Q} \text{ and } x < \pi(n-1), \\ x - \pi(n-1), & \text{if } x \in \mathbf{Q} \text{ and } x \geq \pi(n-1), \\ -x + \pi(n-1), & \text{if } x \notin \mathbf{Q} \text{ and } x \geq \pi(n-1). \end{cases}$$

6. Consider a real-valued continuous function f defined on interval $[a, b]$ with a property that $f(a) = a$ and $f(b) = b$. Let g be any continuous function that maps $[a, b]$ into itself. Prove that there is $x^* \in [a, b]$ such that $f(x^*) = g(x^*)$. Will the statement remain true if g is just continuous on $[a, b]$? Prove or give counter-example.

   Solution. If $f(a) = g(a)$ or $f(b) = g(b)$ then we are done. So suppose not. Then $f(a) \neq g(a)$ and $f(b) \neq g(b)$, which immediately implies $f(a) - g(a) < 0$ and $f(b) - g(b) > 0$. Applying an intermediate value theorem to $h(x) = f(x) - g(x)$ we get the result we seek.

   The statement is clearly false if $g(x)$ is an arbitrary continuous function, say $g(x) = M > \max_{x \in [a,b]} f(x)$.

7. Suppose that $\{f_n\}$ is a sequence of non-decreasing functions that map the unit interval into itself. Suppose that
$$\lim_{n \to +\infty} f_n(x) = f(x)$$
   pointwise and f is a continuous function. Prove that the convergence of $f_n(x)$ to $f(x)$ is uniform, i.e. prove that[5]
$$\forall \varepsilon > 0 \ \exists N_\varepsilon : n > N_\varepsilon \quad |f_n(x) - f(x)| < \varepsilon \text{ for all } x \in [0, 1].$$

[4]Please use standard Euclidean metric, there is no need to be excessively creative.

[5]Like with uniform continuity, same ε works for all x in uniform convergence, whereas in pointwise convergence, ε will, in general, depend on the choice of x.

Solution. To begin, note that we do not require each f_n to be continuous, rather just monotone. However, as you will convince yourself here, monotonicity is a very strong condition (e.g. monotone functions are of *bounded variation*, they have only countably many simple jump discontinuities, etc). Here, monotonicity will allow us to prove a uniform convergence of sequence of (possibly) discontinuous functions to a continuous limit, quite a remarkable result indeed.

Fix $\varepsilon > 0$. To demonstrate a uniform convergence we need to find such $N_\varepsilon \in \mathbf{N}$ that for all $n > N_\varepsilon$ $|f_n(x) - f(x)| < \varepsilon$ and that N_ε works for all $x \in [0, 1]$.

We begin by observing that limiting function $f(x)$ has to be monotone (inequalities are preserved by limits). Thus, we have a continuous monotone function defined on a unit interval and mapping it into itself. Lets partition an image of function $f(x)$ into pairwise disjoint intervals each of the length $\frac{\varepsilon}{2}$.. This partition on the range of $f(x)$ will induce a partition on the domain. Lets denote the endpoints of those intervals by $\{x_0 = 0,\, x_1,\, x_2,\, \ldots,\, x_k,\, x_{k+1},\, \ldots,\, x_{M-1},\, x_M = 1\}$ and call these endpoints "centers." Note that for any $k = 0,\, 1,\, \ldots,\, M - 1$ we have

$$f(x_{k+1}) - f(x_k) < \frac{\varepsilon}{2}$$

by construction.

The idea of the proof is to use the pointwise convergence of $\{f_n(x)\}$ to $f(x)$ at the "centers" to get single N_ε that will work for all $x \in [0,\, 1]$. We then exploit the fact that both $f(x)$ and $f_n(x)$ are non-decreasing to show that $|f(x) - f_n(x)| < \varepsilon$ for all $x \in [x_k,\, x_{k+1}]$.

So, $\{f_n(x_k)\} \to f(x_k)$ for all $k = 0,\, 1,\, \ldots,\, M$ therefore, there exists $N_\varepsilon^{(k)}$ such that

$$|f_n(x_k) - f(x_k)| < \frac{\varepsilon}{2}.$$

Let $N_\varepsilon = \max\left\{N_\varepsilon^{(1)},\, N_\varepsilon^{(2)},\, \ldots,\, N_\varepsilon^{(M)}\right\}$. By monotonicity we get that for all $x \in [x_k,\, x_{k+1}]$

$$\begin{aligned} f_n(x_k) &\le f_n(x) \le f_n(x_{k+1}) \\ f(x_k) &\le \; f(x) \; \le f(x_{k+1}) \end{aligned}$$

Now, lets consider $f_n(x) - f(x)$. Monotonicity yields

$$f_n(x) - f(x) \le f_n(x_{k+1}) - f(x_k).$$

Lets "triangulate" this inequality by going through f at "center" x_{k+1} to get

$$f_n(x_{k+1}) - f(x_k) = f_n(x_{k+1}) - f(x_{k+1}) + f(x_{k+1}) - f(x_k) \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Note that the first $\frac{\varepsilon}{2}$ comes from pointwise convergence at the "center" x_{k+1} and the second $\frac{\varepsilon}{2}$ comes from our construction of the partition.

Finally, lets observe that showing $f_n(x) - f(x) > -\varepsilon$ is analogous since we have a symmetric problem and since our choice of $[x_k,\, x_{k+1}]$ was completely arbitrary, we are done.

8. Let (X, d) be a metric space. Let $\{x_n\}$ and $\{y_n\}$ be two Cauchy sequences in X. Call $\{x_n\}$ and $\{y_n\}$ *Cauchy equivalent* if $x_0, y_0, x_1, y_1, \ldots$ is a Cauchy sequence itself.

   (a) Prove that $\{x_n\}$ and $\{y_n\}$ are Cauchy equivalent iff $\lim_{n\to\infty} d(x_n, y_n) = 0$.

   Solution.
   $(\Longrightarrow)$ Fix $\frac{\varepsilon}{2} > 0$. Since $\{x_n\}$ and $\{y_n\}$ are Cauchy equivalent, there exists N_ε such that whenever $n, m > N_\varepsilon$ we must have $d(x_n, x_m) < \frac{\varepsilon}{2}$ and $d(y_n, y_m) < \frac{\varepsilon}{2}$ and $d(y_n, x_m) < \frac{\varepsilon}{2}$.
   By triangle inequality we have

$$d(x_n, y_n) \leq d(x_n, x_m) + d(x_m, y_n) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

   $(\Longleftarrow)$ Similarly, by triangle inequality

$$d(x_n, y_m) \leq d(x_n, x_m) + d(x_m, y_m) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

   and the desired result follows immediately, once we observe that both $\{x_n\}$ and $\{y_n\}$ are Cauchy.

   (b) Let $\{x_n\}$ and $\{y_n\}$ be two Cauchy equivalent sequences and $\{z_n\}$ another Cauchy sequence. Prove that

$$\lim_{n\to\infty} d(x_n, z_n) = \lim_{n\to\infty} d(y_n, z_n).$$

   Solution. Lets start by observing that

$$d(x_n, z_n) \leq d(x_n, y_n) + d(y_n, z_n)$$

   and

$$d(y_n, z_n) \leq d(y_n, x_n) + d(x_n, z_n).$$

   Taking limits and using the result from part (a) we have

$$\lim_{n\to\infty} d(x_n, z_n) \leq \lim_{n\to\infty} d(y_n, z_n)$$

   and

$$\lim_{n\to\infty} d(y_n, z_n) \leq \lim_{n\to\infty} d(x_n, z_n).$$

   The desired result follows immediately.

   (c) Show that equivalence of Cauchy sequences is an equivalence relation on X.

   Solution. Since reflexivity and symmetry of Cauchy equivalence is trivial, we just show transitivity. But latter follows immediately using the results from parts (a) and (b).

(d) Let X^* be a set of equivalence classes of Cauchy sequences in X. Prove that the function
$$\{x_n\}, \{y_n\} \rightarrow \lim_{n\to\infty} d(x_n, y_n)$$
defines a metric on X^*.

Solution. Non-negativity, equality to zero and symmetry property of the metric are immediate, so it remains to show just triangle inequality. So, let $[\{x_n\}]$ denote an equivalence class containing $\{x_n\}$ and let $[\{y_n\}]$ and $[\{z_n\}]$ be equivalence classes containing $\{y_n\}$ and $\{z_n\}$ respectively. We need to demonstrate that

$$\lim_{n\to\infty} d(x_n, z_n) \leq \lim_{n\to\infty} d(x_n, y_n) + \lim_{n\to\infty} d(y_n, z_n).$$

This result follows immediately from the triangle inequality for d.