

Economics 204  
Fall 2013  
Problem Set 2 Suggested Solutions

1. Show that the sequence  $\{x_n\}$  in a metric space  $X$  converges to  $x$  if and only if every subsequence has  $x$  as a cluster point.

**Solution.** Recall that  $x$  is a cluster point if and only if there is a subsequence that converges to  $x$  (Theorem 2.4 in de la Fuente.) As such, it suffices to show that  $x_n \rightarrow x$  if and only if every subsequence has in turn a subsequence that converges to  $x$ .

( $\Rightarrow$ ) If the sequence converges to  $x$ , then clearly every subsequence converges to  $x$ . To check this let  $x_{n_k}$  be a subsequence. For any  $\epsilon > 0$  convergence of  $x_n$  tells us that  $\exists N(\epsilon)$  such that  $n > N(\epsilon) \Rightarrow d(x_n, x) < \epsilon$ . In particular, for all  $n_k > N(\epsilon)$  we must have  $d(x_{n_k}, x) < \epsilon$ . Hence, the subsequence  $x_{n_k}$  converges to  $x$ . Now note that  $x_{n_k}$  is a subsequence of itself and because it converges to  $x$  we have found a subsequence of  $x_{n_k}$  that converges to  $x$ .

( $\Leftarrow$ ) Assume every subsequence  $x_{n_k}$  itself has a subsequence that converges to  $x$ . We wish to show that this implies the convergence of the full sequence. Proceeding by contradiction, assume not. Then, from the definition, there must be some  $\epsilon > 0$  such that  $d(x_n, x) > \epsilon$  for infinitely many  $n$ . Define  $x_{n_k}$  to then be the (infinite) subsequence of  $x_n$  for which this is true. It then follows that  $x_{n_k}$  itself has no subsequence that converges to  $x$  since every term of the full sequence was chosen to be at least distance  $\epsilon$  from  $x$ . Contradiction.

2. Let  $A \subset \mathbf{R}$  be uncountable. Prove that there is a sequence of distinct points in  $A$  converging to a point of  $A$ .

**Solution.** Let's prove this claim by contradiction. Suppose  $A \subset \mathbf{R}^n$  contains no sequence of distinct points, converging to a point in  $A$ . Then, it must be the case that all points in  $A$  are "isolated," i.e. for every  $a \in A$  there exist some  $\epsilon_a$ -neighbourhood, containing only the point itself  $B_{\epsilon_a} \cap A = \{a\}$ . Now, it is possible that some open balls  $B_{\epsilon_a}$  might "overlap," so pick  $\delta_a = \frac{\epsilon_a}{2}$  and observe that  $\{B_{\delta_a}\}$  collection of balls is disjoint. In each ball we can pick a point with a rational coordinates, therefore, we will have an injection between points of  $A$  and  $\mathbf{N}^{2n}$ . By problem 5 of problem set 1 we know that  $A$  must be countable. Contradiction.

3. Some practice with "relative" openness.

- (a) Given a metric space  $X$ , let  $Y$  be a metric subspace of  $X$ , and take any  $A \subset Y$ . Show that  $A$  is open in  $Y$  if and only if  $A = O \cap Y$  for some open subset  $O$  of  $X$ , and is closed in  $Y$  if and only if  $A = C \cap Y$  for some closed subset  $C$  of  $X$ .

**Solution.** “Open” ( $\Rightarrow$ ) Let  $A$  be open as a subset of  $Y$ . If  $A = \emptyset$  then the problem is trivial and we have nothing to prove as  $A = \emptyset \cap Y$  and an empty set is open subset of  $X$ . So suppose  $A$  is not empty and we need to show that  $A$  is open. By definition of  $A$  being open we can construct an open ball in  $Y$  for all  $a \in A$ ,  $B_\varepsilon^Y(a) = \{b \in Y : d(b, a) < \varepsilon\} \subset A$  (where  $\varepsilon > 0$  clearly depends on point  $a$ , but we ignore that for the ease of notation.) Lets take a union of those balls  $B^Y = \cup_{a \in X} B_\varepsilon^Y(a)$  and note that  $A = B^Y$ . Why? By construction,  $B^Y \subset A$  so lets show that  $A \subset B^Y$ . Take any  $a \in A$ , there must be at least one ball containing it, thus it is in union.

Now, lets expand all those  $\varepsilon$  balls to include also elements in  $X$  that are no further away then  $\varepsilon$  from  $a \in A$ , i.e. lets define  $B_\varepsilon^X(a) = \{b \in X : d(a, b) < \varepsilon\}$ . By construction, each  $B_\varepsilon^X(a)$  is an open set in  $X$ , so their union  $B^X = \cup_{a \in X} B_\varepsilon^X(a)$  is open in  $X$  as well. Finally, observe that  $B_\varepsilon^Y(a) = B_\varepsilon^X(a) \cap Y$  implies that  $B^Y = B^X \cap Y$  because union and intersection distribute. Thus,  $A = B^X \cap Y$ , and we get the result we desire.

( $\Leftarrow$ ) Let  $A = O \cap Y$  for some open subset  $O$  of  $X$ . Again, as before if  $A = \emptyset$  then there is nothing to prove, so suppose  $A \neq \emptyset$ . Pick any  $a \in A$ , and note that  $a \in O \subset X$ , which implies that there exists an open ball  $B_\varepsilon^X(a) \subset O$  in  $X$ , because  $O$  is open. Lets define  $B_\varepsilon^Y(a) = \{b \in Y : d(a, b) < \varepsilon\}$  an open ball in  $Y$  around  $a$  and note that  $B_\varepsilon^Y(a) = B_\varepsilon^X(a) \cap Y$  by construction. Because  $B_\varepsilon^Y(a) \subset O \cap Y$  again by construction, it must be the case that  $B_\varepsilon^Y(a) \subset A$ , so  $A$  is, indeed, open.

“Closed.” ( $\Rightarrow$ ) Let  $A$  be closed as a subset of  $Y$ . Consider an open set  $Y \setminus A$  and note that arguments above imply  $Y \setminus A = O \cap Y$  for some  $O$  open in  $X$ . Observe that

$$\begin{aligned} A &= Y \setminus (Y \setminus A) = Y \setminus (O \cap Y) = (Y \setminus O) \cup (Y \setminus Y) = \\ &= (Y \setminus O) = (X \setminus O) \cap Y = C \cap Y \end{aligned}$$

and since  $C = O^c$  it is a closed set in  $X$ .

( $\Leftarrow$ ) Let  $A = C \cap Y$  for some closed set  $C$  in  $X$  and lets show that complement of  $A$  is open in  $Y$ . Observe that

$$A^c = Y \setminus A = Y \setminus (C \cap Y) = (Y \setminus C) \cup (Y \setminus Y) = (X \setminus C) \cap Y$$

and because  $(X \setminus C)$  is open in  $X$  by arguments given above  $A^c$  is indeed open in  $Y$ . We are done.

(b) Let  $Y$  be open in  $X$ , prove that

$$A \text{ is open in } X \text{ iff } A \text{ is open in } Y.$$

**Solution.**

( $\Leftarrow$ ) Let  $A$  be open in  $Y$ . By part (a) of this exercise,  $A = O \cap Y$  for some open set  $O$  in  $X$ . Thus,  $A$  is an intersection of two open sets in  $X$ , therefore it must be open in  $X$ .

( $\Rightarrow$ ) Let  $A$  be open in  $X$ . Since  $A \subset Y$  it must be the case that  $A = A \cap Y$  thus by part (a) of this exercise,  $A$  is open in  $Y$ .

- (c) Can you give an example of either side of the implication in (b) not holding when  $Y$  is not necessarily open in  $X$ ?

**Solution.** Note that we did not really use openness of  $Y$  in the ( $\Rightarrow$ ) direction. However, for necessity it is crucial. For instance,  $(0, 1) \times \{0\}$  is an open subset of  $[0, 1] \times \{0\}$ , but it is an open subset of  $[0, 1]^2$  (clearly, because the  $[0, 1] \times \{0\}$  is not open in  $[0, 1]^2$ .)

4. Prove the following result

$$\overline{\text{Int } \overline{\text{Int } A}} = \overline{\text{Int } A}$$

**Solution.** We prove it by two-way set inclusion, where one side follows immediately

$$\overline{\text{Int } \overline{\text{Int } A}} \subset \overline{\text{Int } A}.$$

To see this, note that  $\text{Int } \overline{\text{Int } A} \subset \overline{\text{Int } A}$  because  $\text{Int } B \subset B$  for any set  $B$ . Also, by definition of closure  $A \subset B \implies \overline{A} \subset \overline{B}$  (the smallest closed set containing a larger set can't be smaller than the smallest closed set containing a smaller set.) Therefore, we must have

$$\overline{\text{Int } \overline{\text{Int } A}} \subset \overline{\overline{\text{Int } A}}.$$

But the closure of the closure is just the closure itself, and we get the result we desire.

Now, to prove the other direction of set inclusion, note that it suffices to prove that  $\text{Int } \overline{\text{Int } A} \supset \text{Int } A$ . Also, observe that we always have  $\text{Int } A \subset \overline{\text{Int } A}$ . Therefore,  $\text{Int } \overline{\text{Int } A} \subset \text{Int } \overline{\text{Int } A}$ , since by definition of interior  $A \subset B \implies \text{Int } A \subset \text{Int } B$  (the largest open set contained in a larger set can't be smaller than the largest open set contained in a smaller set) and  $\text{Int } \overline{\text{Int } A} = \text{Int } A$  (the largest open set of the largest open set contained in  $A$  is just the largest open set contained in  $A$ .)

5. How many pairwise disjoint sets can one obtain using operators of closure and interior?

**Solution.** At most two. For any set  $A$  we must have  $\text{Int } A \subset A \subset \overline{A}$ . Therefore, the only way to get disjoint sets is for set  $A$  to have an empty interior, e.g.

$\mathbf{Q} \in \mathbf{R}$ . Note that  $\text{Int } Q = \emptyset$  and  $\overline{\mathbf{Q}} = \mathbf{R}$  (and  $\emptyset \notin \mathbf{R}$ ).

Now, another interesting question might be how many pairwise *distinct* sets can one obtain using only operators of closure and interior? Here the answer is six (actually seven, if we include the set itself). Consider the following example

$$A = ((-1, 0) \cap \mathbf{Q}) \cup (0, 1) \cup (1, 2) \cup \{3\}.$$

Then we have

$$\begin{aligned}\text{Int } A &= (0, 1) \cup (1, 2) \\ \overline{\text{Int } A} &= [0, 2] \\ \text{Int } \overline{\text{Int } A} &= (0, 2) \\ \overline{A} &= [-1, 2] \cup \{3\} \\ \text{Int } \overline{A} &= (-1, 2) \\ \overline{\text{Int } \overline{A}} &= [-1, 2]\end{aligned}$$

Why cannot we have more? We know that for any set  $A$  we must have  $\overline{\overline{A}} = \overline{A}$  and  $\text{Int } \text{Int } A = \text{Int } A$ , that is why, we just need to consider the sets that we get from set  $A$  by alternating operations of closure and interior. But, from problem 4 we know that

$$\overline{\overline{\text{Int } \overline{A}}} = \overline{\text{Int } A}.$$

Moreover, we also claim that

$$\text{Int } \overline{\overline{\text{Int } \overline{A}}} = \text{Int } \overline{A}.$$

To see this, observe that  $\text{Int } \overline{A} \subset \overline{\text{Int } \overline{A}}$  because for any set  $A$  we have  $\text{Int } \overline{A} \subset \overline{A}$  which implies  $\text{Int } \overline{A} \subset \overline{\text{Int } \overline{A}}$ . Now, to see that  $\text{Int } \overline{A} \supset \overline{\text{Int } \overline{A}}$  note that  $\overline{A} = \overline{\overline{A}} \supset \overline{\text{Int } \overline{A}}$ . Therefore, only following sets can be pairwise “distinct” (but, clearly, not necessarily):

$$A, \text{Int } A, \overline{\text{Int } A}, \text{Int } \overline{\text{Int } A}, \overline{A}, \text{Int } \overline{A}, \overline{\overline{\text{Int } \overline{A}}}.$$

The example of  $A = \emptyset$  shows that all those sets can be equal. The example at the beginning shows that the maximal number six can be achieved.

## 6. Some practice with continuity

- (a) Use the “pre-image of a closed set is closed” definition of continuity to show that  $S = \{(x, y) \mid x^2 + y^2 \leq 1\} \subset \mathbf{R}^2$  is closed.

**Solution.** Define the function  $f : \mathbf{R}^2 \rightarrow \mathbf{R}$  by  $f(x, y) = x^2 + y^2$ . Note that  $f^{-1}([0, 1]) = S$  and  $[0, 1]$  is closed in  $\mathbf{R}$ . Furthermore, function  $f$  is continuous because it is a sum of two real-valued continuous functions  $f_1(x, y) = x^2$  and  $f_2(x, y) = y^2$ , therefore  $S$  must be closed in  $\mathbf{R}^2$ .

- (b) Suppose that  $f : X \rightarrow Y$  is continuous. If  $x$  is a limit point of  $A \subset X$ , is it necessarily true that  $f(x)$  is a limit point of  $f(A)$ ? (Recall that a limit point of a set  $A \subset X$  is defined as a point  $x \in X$  such that  $B_\epsilon(x)$  contains some element of  $A \setminus \{x\}$  for any  $\epsilon > 0$ .)

**Solution.** Really, the answer depends on which textbook you consult. For us, though, the correct answer — which depends crucially on the definition of “limit point” — is no. A limit point of a set  $A \subset X$  is defined as a point  $x \in X$  such that  $B_\epsilon(x)$  contains some element of  $A \setminus \{x\}$  for any  $\epsilon > 0$ . Notice that this excludes  $x$  itself. So if  $f$  is a constant function then  $f(x)$  is the same, single point for all  $x \in A$ . Because there is no other point in  $f(A)$ , the set  $A \setminus \{f(x)\}$  is empty and it is impossible to satisfy the limit point definition. Other texts allow isolated points in the definition of limit point.

7. Let  $f : \mathbf{R} \rightarrow \mathbf{R}$  be continuous function such that  $|f(x) - f(x')| \geq |x - x'|$  for all  $x$  and  $x'$ . Prove that the range of  $f$  is all of  $\mathbf{R}$ .

**Solution.** We prove the result by showing that range of  $f$  must be both open and closed and, thus, range of  $f$  is all of  $\mathbf{R}$ .

First, let's show that range of  $f$  is open. Our inequality immediately implies that  $f$  must be injective. Together with continuity of  $f$ , Intermediate Value Theorem implies that  $f$  must be strictly monotonic (Can you see why? Hint: proof by contradiction.) Strong monotonicity and continuity of  $f : \mathbf{R} \rightarrow \mathbf{R}$  means that  $f$  must map open sets to open sets (Why? You might want to use the fact that intervals are the only connected sets in  $\mathbf{R}$  and continuous function must map connected set into connected set.) So we have just shown that range of  $f$  is open.

Now, let's show that range of  $f$  is closed by demonstrating that it contains all its limit point. To this end, take a sequence  $\{y_n\} = f(x_n)$  in  $f(\mathbf{R})$  that converges to some  $y \in \mathbf{R}$ . Such  $\{y_n\}$  must be Cauchy, and our inequality implies that  $\{x_n\}$  is Cauchy as well. Suppose that  $\lim x_n = x$ . By continuity of  $f$  we have that

$$f(x) = f(\lim x_n) = \lim f(x_n) = y$$

Thus,  $y$  is in the range of  $f$  and, therefore,  $f(\mathbf{R})$  is closed. We get the result we desire.

8. Let  $(X, \rho)$  and  $(Y, \sigma)$  be metric spaces. Let  $\{f_n\}$  be a sequence of bijective functions from  $X$  to  $Y$  and  $\{g_n\}$  be the sequence of their uniformly continuous inverses. Prove that uniform convergence of  $f_n \rightarrow f$  implies uniform convergence of  $g_n \rightarrow g$ , where  $g$  is a uniformly continuous inverse of  $f$ .

**Solution.** First, since uniform convergence of functions was not properly introduced either in lecture or in section, let's say a few words about it here. Last

couple of lectures examined thoroughly convergence for sequences of *points* in metric spaces. This exercise is about convergence of sequences of *functions*  $\{f_n(\cdot)\}$  from one metric space  $(X, \rho)$  to another  $(Y, \sigma)$ . So what does it mean for a sequence of functions to converge to a limiting function  $f : X \rightarrow Y$ ? There are several different concepts for convergence of functions with two most important being *pointwise* and *uniform* convergence (among other types of convergence are convergence in  $L^p$ -norm, convergence in measure, etc.)

The former, pointwise convergence, is perhaps the easiest to understand — it is convergence at each point of a domain  $x_0 \in X$ , i.e. fixing a point in domain we obtain a sequence of points  $\{f_n(x_0)\}$  and we know how to work with those. The problem is that this type of convergence is extremely *weak*, i.e. a lot of nice properties of limits are not preserved by this type of convergence. The most important example of that would be that the pointwise limit of continuous functions need not be a continuous function itself, i.e. pointwise convergence does not preserve continuity. For instance, take  $f_n(x) = x^n$  on a unit interval (the limiting function  $f$  is zero except at  $x = 1$ .) Also, pointwise convergence does not preserve either limits or integrals, for instance limits of sequences of functions  $\{f_n(x)\}$  and limits of points in domain  $\{x_n\}$  can't be interchanged.

With pointwise convergence being of a too weak concept to be of much use, uniform convergence addresses most of those issues. The relation between pointwise and uniform convergence is similar to one of continuity and uniform continuity. In the latter, a single  $\delta_\varepsilon$  works for all points in domain, in the former, a single  $N_\varepsilon$ . To give a precise definition, we say that

A sequence of functions  $\{f_n(x)\}$  from one metric space  $(X, \rho)$  to another  $(Y, \sigma)$  **converges uniformly** to a function  $f : X \rightarrow Y$  if for every  $\varepsilon > 0$  there exists  $N_\varepsilon \in \mathbf{N}$  such that  $\sigma(f_n(x), f(x)) < \varepsilon$  for every  $n > N_\varepsilon$  and  $x \in X$ . The function  $f$  is the **uniform limit** of the sequence of functions  $\{f_n(x)\}$ .

Note that uniform limit is frequently denoted  $\{f_n(x)\} \rightrightarrows f$ .

Now, let's prove our result. The key to solving this problem is to realize that uniformly continuous maps preserve uniform convergence of function. Thus, to prove our desired result we need to prove this claim and show that

$$\sigma(f(g(y)), f(g_n(y))) \rightrightarrows 0$$

as  $n \rightarrow \infty$  since  $g$  is uniformly continuous function by assumption. So, let's proceed in two steps: firstly, we will prove that  $f(g_n)$  converges uniformly, and, secondly, we prove our claim that uniform convergence is preserved under uniformly continuous maps.

First step. Let's fix  $\varepsilon > 0$ , and note that since  $f_n \rightrightarrows f$ , there exists  $N_\varepsilon \in \mathbf{N}$  such that  $\sigma(f_n(x), f(x)) < \varepsilon$  for all  $n > N_\varepsilon$  and  $x \in X$ . We want to show that  $f(g_n) \rightrightarrows f(g)$  or, equivalently, since  $f(g) = id_Y$  that  $f(g_n) \rightrightarrows id_Y$ . So, recalling

that  $f$  is bijective set  $x = g_n(y) \iff f_n(x) = y$ , we have for  $n$  sufficiently large

$$\sigma(f_n(g_n(y)), f(g_n(y))) < \varepsilon \iff \sigma(y, f(g_n(y))) < \varepsilon.$$

Since  $y \in Y$  was arbitrary, we have that  $f(g_n) \rightrightarrows id_Y$ .

Second step. Now, let's show that a uniformly continuous map preserves uniform continuity, i.e. if  $g : Y \rightarrow X$  is uniformly continuous map and  $\{h_n(x)\}$  is any sequence such that  $h_n(x) \rightrightarrows h(x)$ , then  $g(h_n(x)) \rightrightarrows g(h(x))$ . To see this, fix an  $\varepsilon > 0$  and by uniform continuity of  $g$  get  $\delta_\varepsilon > 0$  such that  $\rho(g(y_1), g(y_2)) < \varepsilon$  whenever  $\sigma(y_1, y_2) < \delta_\varepsilon$  for all  $y_1, y_2 \in Y$ . Now, use uniform convergence of  $\{h_n(x)\}$  to pick  $n_\delta \in \mathbf{N}$  sufficiently large to get  $\rho(h_n(x), h(x)) < \delta_\varepsilon$  for any  $x \in X$ . Therefore, it must be the case that  $\rho(g(h_n(x)), g(h(x))) < \varepsilon$  and since  $x$  was arbitrary, we get our result  $g(h_n(x)) \rightrightarrows g(h(x))$ .

Finally, setting  $h = f(g_n)$  and by the reasoning given above we have

$$g(f(g_n)) \rightrightarrows g(f(g)) \iff g \circ (f \circ g_n) \rightrightarrows g \circ (f \circ g) \iff (g \circ f) \circ g_n \rightrightarrows (g \circ f) \circ g.$$

Thus, we get the result we desire that  $g_n \rightrightarrows g$ .

9. Show that if  $x_n$  and  $y_n$  are Cauchy sequences from a metric space  $X$ , then  $d(x_n, y_n)$  converges.

**Solution.** Because  $X$  is not necessarily complete, we cannot rely on the convergence of  $x_n$  and  $y_n$ . The fact that the sequences are Cauchy means that for all  $\varepsilon > 0$ , there exists an  $N_x(\varepsilon)$  such that for all  $m, n \geq N_x(\varepsilon) \Rightarrow d(x_m, x_n) < \varepsilon$  and there exists an  $N_y(\varepsilon)$  such that for all  $m, n \geq N_y(\varepsilon) \Rightarrow d(y_m, y_n) < \varepsilon$ . We will use this to show that the sequence  $d(x_n, y_n)$  is Cauchy, and because  $\mathbf{R}$  is complete it must converge.

First let us make note of two facts which come from repeated application of the triangle inequality:

- $d(x_n, y_n) \leq d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n)$
- $d(x_m, y_m) \leq d(x_m, x_n) + d(x_n, y_n) + d(y_n, y_m)$

Rearranging these (by isolating the expression  $d(x_m, y_m) - d(x_n, y_n)$ ) yields

$$-(d(x_m, x_n) + d(y_m, y_n)) \leq d(x_m, y_m) - d(x_n, y_n) \leq d(x_m, x_n) + d(y_m, y_n),$$

or  $|d(x_m, y_m) - d(x_n, y_n)| \leq d(x_m, x_n) + d(y_m, y_n)$ . Now given  $\varepsilon > 0$ , choose  $N(\varepsilon) > \max\{N_x(\frac{\varepsilon}{2}), N_y(\frac{\varepsilon}{2})\}$ . Then  $n \geq N(\varepsilon) \Rightarrow |d(x_m, y_m) - d(x_n, y_n)| \leq d(x_m, x_n) + d(y_m, y_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ . So  $d(x_n, y_n)$  is Cauchy and consequently converges.