Econ 204 – Problem Set 2

Due Tuesday, August 5

1. Prove that a convergent sequence in an arbitrary metric space (X, d) has exactly one cluster point.

   Solution The limit of a convergent sequence is a cluster point so a convergent sequence must have at least one cluster point. We will prove that if c is a cluster point, the sequence cannot converge to a different point. It follows then, that a sequence with more than one cluster point cannot converge to any point, because each cluster point excludes all other points from being the limit. Thus, a convergent sequence must have exactly one cluster point.

   Let $\{x_n\}$ be a convergent sequence in a metric space (X, d), let c be a cluster point, and consider any $x \in X$ such that $d(x, c) > 0$. We will show that since c is a cluster point, there will always be an element of $\{x_n\}$ within ϵ distance of c. However, since x and c are distinct, this point cannot be arbitrarily close to x, so the sequence does not converge to x. Given any $\epsilon > 0$, $\forall N \in \mathbb{N}$, $\exists n > N$ such that $d(c, x_n) < \epsilon$, because c is a cluster point. For $\epsilon < \frac{d(c,x)}{2}$ and any value of n that satisfies the above inequality, we have

   $$d(c, x_n) < \epsilon < \frac{d(c, x)}{2}$$

   or

   $$d(c, x_n) + \frac{d(c, x)}{2} < d(c, x).$$

   Rearranging, we have

   $$\frac{d(c, x)}{2} < d(c, x) - d(c, x_n)$$

   and by construction and the triangle equality

   $$\epsilon < \frac{d(P, x)}{2} < d(c, x) - d(c, x_n) \leq d(c, x_n).$$

   So for any small ϵ and any $N \in \mathbb{N}$, we can always find a point later than N in the sequence within ϵ of c, which means that it is more than ϵ away from x. Thus, $\{x_n\}$ cannot converge to x. This means that any sequence that has more than one cluster point cannot converge to any point, so a convergent sequence has exactly one cluster point.

2. The decimal expansion of $\frac{1}{7}$ is $0.142857142857142857\ldots$ etc. repeating forever. Suppose we construct the sequence $\{x_n\}$ by, for each n, x_n is the n^{th} decimal place in the infinite expansion of $\frac{1}{7}$. Prove that every sequence made up of the elements from the set $Y = \{1; 4; 2; 8; 5; 7\}$ is a subsequence of $\{x_n\}$.

Solution We will prove this by construction. In the sequence $\{x_n\}$, each $y \in Y$ is repeated an infinite number of times. Put formally, $\forall y \in Y$, $\forall N \in \mathbb{N}$, $\exists m > N$ such that $x_m = y$.

Now let $\{y_n\}$ be a sequence of elements of Y. We will use induction to show that there we can construct a strictly increasing function $f : \mathbb{N} \to \mathbb{N}$ such that $y_n = x_{f(n)}$. First, we know from above that for y_1, $\exists m > 1$ such that $x_m = y_1 \in Y$. So we can define $f(1) = m$ and we have $y_1 = x_m = x_{f(n)}$. Now suppose that we can define $f(n)$ such that $y_n = x_{f(n)}$. To show that we can define $f(n+1)$, we again refer to the fact that $\exists m > f(n)$ such that $x_m = y_{n+1}$ and we define $f(n+1) = m$ for such an m. This guarantees that $y_{n+1} = x_m = f_{n+1}$.

Since f is strictly increasing and $y_n = x_{f(n)}$ for all $n \in \mathbb{N}$ we know that $\{y_n\}$ is a subsequence of $\{x_n\}$.

3. Show whether the following are open, closed, both, or neither:

   (a) The interval $(0, 1)$ as a subset of $\mathbb{R}$.

   Solution Open, not closed eg. $\frac{1}{n} \to 0$.

   (b) The interval $(0, 1)$ imbedded in $\mathbb{R}^2$ as the subset $\{(x, 0) : x \in (0, 1)\}$.

   Solution Not open, since it's boundary is not empty; and not closed, since the origin is a limit point not in the set.

   (c) $\mathbb{R}$ as a subset of $\mathbb{R}$.

   Solution Open and closed.

   (d) $\mathbb{R}$ imbedded in $\mathbb{R}^2$ as the subset $\{(x, 0) : x \in \mathbb{R}\}$.

   Solution Not open and closed, since $\mathbb{R}^2 \setminus \mathbb{R}$ is open and not closed, the origin is a limit point.

   (e) $\{(x, y, z) : 0 \leq x + y \leq 1, z = 0\}$ as a subset of $\mathbb{R}^3$

   Solution Not open, closed.

   (f) $\{(x, y) : 0 < x + y < 1\}\}$ as a subset of $\mathbb{R}^2$

   Solution Open, not closed.

(g) $\{\frac{1}{n} : n \in \mathbb{N}\}$ as a subset of $\mathbb{R}$

Solution Not open, not closed.

(h) $\{\frac{1}{n} : n \in \mathbb{N}\}$ as a subset of the interval $(0, \infty)$

Solution Not open, closed since there are no convergent sequences in it.

4. Let A be a subset of a metric space. Prove that $\text{int}(\text{int}(A)) = \text{int}(A)$.

   Solution Couple of ways to do this one; one is the standard set-equality proof: $\text{int}(\text{int}(A))$ is the largest open set contained within $\text{int}(A)$. But $\text{int}(A)$ is itself open; after all, it's the largest open set contained in A. Thus $\text{int}(\text{int}(A)) = \text{int}(A)$.

5. Let X denote the set of all bounded infinite sequences of real numbers $\{a_n\}_{n=1}^{\infty}$ (hereafter denoted simply as a_n). Define the "distance" between two sequences a_n and b_n to be: $d(a_n, b_n) = \sum_{n=1}^{\infty} 2^{-n}|a_n - b_n|$. Show that (X, d) is a metric space.

   Solution Elements of X are sequences of numbers. Put $\{a_n\}, \{b_n\}, \{c_n\} \in X$. We check only the triangle inequality as the other two properties of a metric are obviously satisfied: $d(\{a_n\}, \{b_n\}) + d(\{b_n\}, \{c_n\}) = \sum_{n=1}^{\infty} 2^{-n}|a_n - b_n| + \sum_{n=1}^{\infty} 2^{-n}|b_n - c_n| = \sum_{n=1}^{\infty} 2^{-n}(|a_n - b_n| + |b_n - c_n|) \geq \sum_{n=1}^{\infty} 2^{-n}|a_n - c_n| = d(\{a_n\}, \{c_n\})$, where the last inequality follows from the usual triangle inequality for the real numbers.