Economics 204, Fall 2012
Problem Set 3
Due Friday, August 3 in Lecture

1. Give an example of each of the following (and prove that your example indeed works):
   (a) A complete metric space that is bounded but not compact.
   (b) A metric space none of whose closed balls are complete.

2. Let (X, d) be a metric space.
   (a) Suppose that for some $\varepsilon > 0$, every open ε-ball in X has compact closure. Show that X is complete.
   (b) Suppose that for each $x \in X$ there exists some $\varepsilon > 0$ such that $B_\varepsilon(x)$ has compact closure. Show that X need not be complete.

3. Show that a metric space (X, d) is compact if and only if every infinite subset $S \subseteq X$ has a limit point.[1] Use the open-cover definition of compactness to prove the 'only if' part.

4. Prove or give a counter-example (you don't need to prove that the sets from your counterexamples are connected/disconnected) for each of the following claims:
   (a) The interior of a connected set is connected.
   (b) The closure of a connected set is connected.
   (c) The interior of a disconnected set (i.e. a set that is not connected) is disconnected.
   (d) The closure of a disconnected set is disconnected.

5. Suppose $\Gamma : X \to 2^Y$ is an upper hemicontinuous, compact-valued correspondence, where $X \subseteq \mathbb{R}^n, Y \subseteq \mathbb{R}^m$ for some n, m. Show directly from the definition of upper hemicontinuity that $\Gamma(K) = \bigcup_{x \in K} \Gamma(x)$ is a compact subset of Y for every compact subset $K \subseteq X$.

6. Prove that if the graph of a correspondence is open, then the correspondence is lower hemicontinuous.

[1] Recall from Problem Set 2 that x is a limit point of the set S in a metric space (X, d) iff every open ball around x contains at least one element of S distinct from x. Note that a limit point of a set need not be contained in the set itself!