Economics 204
Fall 2013
Problem Set 3
Due Friday, August 9 in Lecture

1. Some practice with compactness:
   (a) Use the open cover definition of compactness to show that the subset $\left\{\frac{n}{n^2+1},\, n = 0, 1, 2 \ldots\right\}$ of $\mathbf{R}$ is compact.
   (b) Let $O_1 \subset O_2 \subset O_3 \subset \ldots.$ be open subsets of $\mathbf{R}$ with non-empty and bounded complement. Prove that
   $$\bigcup_{j=0}^{\infty} O_j \neq \mathbf{R}.$$
   (c) Provide an example of a decreasing sequence of closed subsets of $\mathbf{R}$ (e.g. $S_1 \supset S_2 \supset S_3 \supset \ldots.$) such that $\bigcap_{n=1}^{\infty} S_n = \emptyset$.

2. Some practice with compactness and completeness:
   (a) Let (X, d) be a metric space. Suppose that for some $\varepsilon > 0$ every $\varepsilon-$ball $B_\varepsilon(x)$ in X has compact closure. Show that X is complete.
   (b) Continue to assume that (X, d) is a metric space. Now, suppose that for each $x \in X$ there is an $\varepsilon > 0$ such that $B_\varepsilon(x)$ has compact closure. Will X still be complete? Prove or give counter-example.

3. Show that a metric space which has countably many points is connected if and only if it contains only one point. Hint: You can use (without proof) the fact that for any $a > 0$, the interval $[0, a]$ in $\mathbf{R}$ is uncountable.

4. Let (X, d) and (Y, ρ) be metric spaces and let A be any subset of Y. Prove that the constant correspondence $\Phi : X \to 2^Y$ defined by $\Phi(x) = A$ for all $x \in X$ is continuous.

5. Let (X, d) be a compact metric space and let $\Psi(x) : X \to 2^X$ be a upper-hemicontinuous, compact-valued correspondence, such that $\Psi(x) : X \to 2^X$ is non-empty for every $x \in X$. Prove that there exists a compact non-empty subset K of X, such that $\Psi(K) = \cup\{\Psi(x) : x \in K\} = K$.

6. Let (X, d) be a complete metric space and $\{T_n\}$ be a sequence of contractive self-maps on X such that $\sup\{\beta_m : m \in \mathbf{N}\} < 1$, where β_m is a contraction modulus of T_m, $m = 1, 2, \ldots$. By the Contraction Mapping Fixed Point Theorem, T_m has a unique fixed point, say x_m. Show that if
$$\sup\{d(T_m(x), T(x)) : x \in X\} \to 0$$

for some $T : X \to X$, then T is a contraction with a unique fixed point $\lim x_m$. It is possible to weaken our assumption and just require that $d(T_m(x), T(x)) \to 0$ for every $x \in X$?