Econ 204 – Problem Set 3

Due Friday, August 8, 2014

1. Call a metric space (X, d) *discrete* if every subset $A \subset X$ is open.[1] Prove or provide a counterexample: every discrete metric space is complete.

2. A function $f : X \to Y$ is *open* if for every open set $A \subset X$, its image $f(A)$ is also open. Show that any continuous open function from $\mathbb{R}$ into $\mathbb{R}$ (with the usual metric) is strictly monotonic.

3. Suppose f, g are continuous functions from metric spaces (X, d) into (Y, ρ). Let E be a dense subset of X (in a metric space, E is dense in X if $\bar{E} = X$). Show that $f(E)$ is dense in $f(X)$. Further, if $f(x) = g(x)$ for every $x \in E$, then $f(x) = g(x)$ for every $x \in X$.

4. Let (X, d) be a metric space.

   (a) Suppose that for some $\varepsilon > 0$, every ε-ball in X has compact closure. Show that X is complete.

   (b) Suppose that for each $x \in X$ there is an $\varepsilon > 0$ such that $B_\varepsilon(x)$ has compact closure. Show by means of an example that X need not be complete.

5. Let (X, d) be a compact metric space and let $\Phi(x) : X \to 2^X$ be a upper-hemicontinuous, compact-valued correspondence, such that $\Phi(x)$ is non-empty for every $x \in X$. Prove that there exists a compact non-empty subset K of X, such that $\Phi(K) \equiv \bigcup_{x \in K} \Phi(x) = K$.

6. Define the correspondence $\Gamma : [0, 1] \to 2^{[0,1]}$ by:

$$\Gamma(x) = \begin{cases} [0,1] \cap \mathbb{Q} & \text{if } x \in [0,1] \backslash \mathbb{Q} \\ [0,1] \backslash \mathbb{Q} & \text{if } x \in [0,1] \cap \mathbb{Q} \end{cases}.$$

   Show that Γ is not continuous, but it is lower-hemicontinuous. Is Γ upper-hemicontinuous at any rational? At any irrational? Does this correspondence have a closed graph?

[1] Notice that any set equipped with the discrete metric forms a discrete metric space, but not every discrete metric space necessarily has the discrete metric!