Economics 204
Fall 2012
Problem Set 3 Suggested Solutions

1. Give an example of each of the following (and prove that your example indeed works):

   (a) A complete metric space that is bounded but not compact.

   (b) A metric space none of whose closed balls are complete.

Solution:

(a) One example of a complete metric space that is bounded but not compact is $\mathbb{R}$ with the discrete metric (i.e. $d(x, y) = 1$ if $x \neq y$ and $d(x, y) = 0$ if $x = y$). This metric space is complete because since $d(x, y) \in \{0, 1\}$ for all $x, y \in \mathbb{R}$ then any Cauchy sequence must eventually be constant (if $d(x, y) < 1$ then $x = y$) and therefore converges. The metric space is bounded since $d(x, y) \leq 1$ for all $x, y \in \mathbb{R}$. Finally, to see that it is not compact consider the open cover $\{B_{1/2}(x)\}_{x \in \mathbb{R}}$. Each of these open balls contains only the point it is centered on and therefore, since $\mathbb{R}$ is not finite, the open cover does not have a finite subcover.

(b) One example of a metric space none of whose closed balls are complete is $\mathbb{Q}$ with the usual Euclidean metric. To see that, start by fixing arbitrary $q \in \mathbb{Q}$ and $\varepsilon > 0$. To differentiate between balls in $\mathbb{Q}$ and in $\mathbb{R}$, we'll use the corresponding subscript. So consider the closed ball $B^{\mathbb{Q}}_{\varepsilon}[q]$ in $\mathbb{Q}$. Note that $B^{\mathbb{Q}}_{\varepsilon}[q] = B^{\mathbb{R}}_{\varepsilon}[q] \cap \mathbb{Q}$ and $B^{\mathbb{Q}}_{\varepsilon}(q) = B^{\mathbb{R}}_{\varepsilon}(q) \cap \mathbb{Q}$.

We will take advantage of the fact that both the rational numbers and the irrational numbers are dense on the real line. In particular, we will use the well-known property that every open set in $\mathbb{R}$ contains infinitely many rational and infinitely many irrational numbers. This allows us to pick some irrational $p \in B^{\mathbb{R}}_{\varepsilon}(q) \subseteq B^{\mathbb{Q}}_{\varepsilon}[q]$. Notice that, as an irrational number, p is not in $B^{\mathbb{Q}}_{\varepsilon}[q]$. The same property allows us to choose some rational $q_n \in B^{\mathbb{R}}_{1/n}(p) \cap$

$B_\varepsilon^{\mathbb{R}}(q)$ for all $n \in \mathbb{N}$. More specifically, we can do that because both $B_{1/n}^{\mathbb{R}}(p)$ and $B_\varepsilon^{\mathbb{R}}(q)$ are open in $\mathbb{R}$ and so is their intersection. Since q_n is rational and, as noted above, $B_\varepsilon^{\mathbb{Q}}(q) = B_\varepsilon^{\mathbb{R}}(q) \cap \mathbb{Q}$, then the sequence $\{q_n\}$ is entirely contained in $B_\varepsilon^{\mathbb{Q}}(q) \subseteq B_\varepsilon^{\mathbb{Q}}[q]$. This sequence also converges in $\mathbb{R}$. To see that, observe that for any $\varepsilon > 0$ we can use the Archimedean property to find some $N \in \mathbb{N}$ such that $\varepsilon > 1/N$. But by the way we chose $\{q_n\}$, we know that $|q_n - p| < 1/n \leq 1/N < \varepsilon$ for all $n \geq N$ and hence $\{q_n\} \to p$. Since this sequence converges, it is necessarily Cauchy with respect to the Euclidean metric. Hence it is Cauchy in our metric space counterexample ($\mathbb{Q}$ with the Euclidean metric). However, it does not converge there since p - its limit in $\mathbb{R}$ - is not in $B_\varepsilon^{\mathbb{Q}}[q]$. Thus the arbitrary closed ball $B_\varepsilon^{\mathbb{Q}}[q]$ in our metric space is not complete.

2. Let (X, d) be a metric space.

   (a) Suppose that for some $\varepsilon > 0$, every open ε-ball in X has compact closure. Show that X is complete.

   (b) Suppose that for each $x \in X$ there exists some $\varepsilon > 0$ such that $B_\varepsilon(x)$ has compact closure. Show that X need not be complete.

Solution:

(a) Let $\varepsilon > 0$ be such that every ε-ball in X has compact closure and let $\{x_n\}$ be a Cauchy sequence in X. We know that there exists some N such that for all $m, n \geq N$ we have $d(x_m, x_n) < \varepsilon$. Consider $B_\varepsilon(x_N)$ and its compact (by assumption) closure, $\overline{B_\varepsilon(x_N)}$.[1] Since $d(x_N, x_m) < \varepsilon$ for all $m \geq N$ we have $x_n \in B_\varepsilon(x_N) \subset \overline{B_\varepsilon(x_N)}$ for all $n \geq N$. The subsequence of $\{x_n\}$ consisting of all x_n such that $n \geq N$ is itself clearly a Cauchy sequence and is contained entirely in $\overline{B_\varepsilon(x_N)}$, which, by hypothesis, is compact. By sequential compactness, a sequence in a compact set must have a convergent subsequence, and Theorem 7.8 in de la Fuente establishes that a Cauchy sequence with a convergent subsequence must itself converge. Thus, the sequence contained in $\overline{B_\varepsilon(x_N)}$ must converge

[1] Despite appearance, the closure of an open ε-ball need not be the corresponding closed ball.

and therefore the Cauchy sequence $\{x_n\}$ must converge as well. Therefore, X is complete.

(b) Let $X = (0, \infty)$. This set has the property that for each x there exists an $\varepsilon > 0$ such that $B_\varepsilon(x)$ has compact closure. For example, given some $x \in X$ we can choose $\varepsilon = x/2$. Then the closure of $B_\varepsilon(x) = \left(\frac{x}{2}, \frac{3x}{2}\right)$ is $\left[\frac{x}{2}, \frac{3x}{2}\right]$, which is a closed and bounded subset of the reals and is therefore compact. However, this space is not a complete metric space because the Cauchy sequence $x_n = 1/n$ does not converge.

3. Show that a metric space (X, d) is compact if and only if every infinite subset $S \subseteq X$ has a limit point.[2] Use the open-cover definition of compactness to prove the 'only if' part.

Solution: $\Rightarrow$: Let (X, d) be a compact metric space. To show that all its infinite subsets have a limit point, we will prove the contrapositive - if A has no limit points, then A must be finite.

So suppose A has no limit points. We start by showing that this implies that $X \setminus A$ must be open. Since A doesn't have any limit points, this implies that for all $x \in X \setminus A$ we can find some open ball $B_x \ni x$ such that $B_x \cap A = \varnothing$ or, equivalently, $B_x \subseteq X \setminus A$. But then the set $\bigcup_{x \in X \setminus A} B_x$ is also contained in $X \setminus A$, while it clearly contains every $x \in X \setminus A$. So $X \setminus A = \bigcup_{x \in X \setminus A} B_x$ and thus $X \setminus A$ must be open as a union of open balls.

Additionally, since no $a \in A$ is a limit point of A we can find open balls $\{U_a\}_{a \in A}$ such that $U_a \cap A = \{a\}$ for all $a \in A$. Note that $\{X \setminus A\} \cup \{U_a\}_{a \in A}$ must then be an open cover of X. Since X is compact, that cover has a finite subcover which must clearly also cover A. Since $X \setminus A$ is disjoint from A, this implies that A must be covered by finitely many of the sets U_a. But each of these sets contains only one element of A. Thus A is a finite set.

$\Leftarrow$: We will show that the property that every infinite set has a limit point implies sequential compactness of X. Start with some sequence

[2] Recall from Problem Set 2 that x is a limit point of the set S in a metric space (X, d) iff every open ball around x contains at least one element of S distinct from x. Note that a limit point of a set need not be contained in the set itself!

$\{x_n\}$ in X and consider the set $A = \{x_n : n \in \mathbb{N}\}$ (i.e. A is the set of all elements of the sequence $\{x_n\}$). We want to show that $\{x_n\}$ has a convergent subsequence. If A is finite, this would imply that there is some $x \in A$ such $x = x_n$ for infinitely many values of n. In such a case, the sequence $\{x_n\}$ has a constant subsequence, which converges trivially.

Assume instead that A is infinite. This implies that A has a limit point. Let that be x. Notice that $B_\varepsilon(x) \cap A$ must be infinite for all $\varepsilon > 0$. It is non-empty, since x is a limit point of A. So if $a_1 \in B_\varepsilon(x) \cap A$, then we can find some $a_2 \in B_{d(x,a_1)}(x) \cap A \subseteq B_\varepsilon(x) \cap A$ that is different from a_1 since $d(x, a_2) < d(x, a_1)$. In this manner, we can inductively find infinitely many distinct elements of $B_\varepsilon(x) \cap A$.

Now let n_1 be such that $x_{n_1} \in B_1(x)$. Since $B_{1/2}(x) \cap A$ is infinite by the above, there must be some $n_2 > n_1$ such that $x_{n_2} \in B_{1/2}(x)$. (If such a n_2 doesn't exist, we would have $B_{1/2}(x) \cap A \subseteq \{x_1, \ldots, x_{n_1}\}$, which contradicts the fact that $B_{1/2}(x) \cap A$ is infinite.) In this manner we can inductively construct a subsequence $\{x_{n_k}\}$ that converges to x (showing that this subsequence indeed converges to x would be analogous to the way we showed that the sequence $\{q_n\}$ converges to p in 1.(b)).

4. Prove or give a counter-example (you don't need to prove that the sets from your counterexamples are connected/disconnected) for each of the following claims:

   (a) The interior of a connected set is connected.

   (b) The closure of a connected set is connected.

   (c) The interior of a disconnected set (i.e. a set that is not connected) is disconnected.

   (d) The closure of a disconnected set is disconnected.

Solution:

(a) The interior of a connected set is not necessarily connected. Consider the following subset of $\mathbb{R}^2$:

$$A = B_1[(-1, 0)] \cup B_1[(1, 0)].$$

The set A is connected (the two closed balls are tangent at $(0,0)$) but its interior is

$$A^o = B_1((-1,0)) \cup B_1((1,0)),$$

which is disconnected (you can verify that the two open balls in this union are separated sets).

(b) The closure of a connected set is indeed connected. Let us prove this by contraposition. Let S be a set with a disconnected closure. We will show that S must also be disconnected. Since $\overline{S}$ is disconnected, we can express it as $\overline{S} = A \cup B$, where A and B are separated. I.e. $\overline{A} \cap B = A \cap \overline{B} = \varnothing$.

Define $A_1 = A \cap S$ and $B_1 = B \cap S$. Since $S \subseteq \overline{S} = A \cup B$, then clearly $S = A_1 \cup B_1$. To show that S is disconnected, it suffices to show that A_1 and B_1 are separated. Notice that since $A \subseteq \overline{A}$ and $S \subseteq \overline{S}$, then $A_1 = A \cap S \subseteq \overline{A} \cap \overline{S}$. But then $\overline{A} \cap \overline{S}$ is a closed set containing A_1 and therefore it also contains $\overline{A_1}$.

Now note that since $\overline{A_1} \subseteq \overline{A} \cap \overline{S} \subseteq \overline{A}$ and $B_1 = B \cap S \subseteq B$, we have

$$\overline{A_1} \cap B_1 \subseteq \overline{A} \cap B = \varnothing$$

and therefore $\overline{A_1} \cap B_1 = \varnothing$. Analogously $A_1 \cap \overline{B_1} = \varnothing$. Therefore A_1 and B_1 are separated and S is disconnected.

(c) The interior of a disconnected set is not necessarily disconnected. Consider the following subset of $\mathbb{R}^2$:

$$B = B_1((0,0)) \cup \{(2,2)\}.$$

The set B is disconnected since $B_1((0,0)) \cap \overline{\{(2,2)\}} = \{(2,2)\} \cap \overline{B_1((0,0))} = \varnothing$. However, the interior of B is

$$B^o = B_1((0,0)),$$

which is a connected set.

(d) The closure of a disconnected set is not necessarily disconnected. Consider again the counterexample from part (a). The set A^o is disconnected but its closure, the set A, is connected.

5. Suppose $\Gamma : X \to 2^Y$ is an upper hemicontinuous, compact-valued correspondence, where $X \subseteq \mathbb{R}^n, Y \subseteq \mathbb{R}^m$ for some n, m. Show directly from the definition of upper hemicontinuity that $\Gamma(K) = \bigcup_{x \in K} \Gamma(x)$ is a compact subset of Y for every compact subset $K \subseteq X$.

   Solution: Let $\{U_\lambda\}_{\lambda \in \Lambda}$ be an open cover of $\Gamma(K)$. The same collection, $\{U_\lambda\}_{\lambda \in \Lambda}$, is also an open cover for $\Gamma(x) \subseteq \Gamma(K)$ for each $x \in K$. Since Γ is compact-valued, there is a finite subcover $\{U_x^1, \ldots, U_x^{n_x}\}$ of $\Gamma(x)$ for each $x \in K$. Let $V_x = U_x^1 \cup \cdots \cup U_x^{n_x}$ for each $x \in K$. Note that V_x is open as the union of open sets.

   Define $\Gamma^{-1}(V_x) = \{y \in X : \Gamma(y) \subseteq V_x\}$. Since Γ is upper hemicontinuous and V_x is open, then we can find an open ball centered at y for any $y \in \Gamma^{-1}(V_x)$ such that the ball is entirely contained in $\Gamma^{-1}(V_x)$. (This is just a restatement of the definition of upper hemicontinuity.) Therefore $\Gamma^{-1}(V_x)$ is open for all $x \in K$. Note that $\Gamma(x) \subseteq V_x$ since V_x is just the union of the elements of a cover of $\Gamma(x)$. Thus $x \in \Gamma^{-1}(V_x)$ for all $x \in K$ and so $\{\Gamma^{-1}(V_x)\}_{x \in K}$ is a cover of K.

   But K is compact so $\{\Gamma^{-1}(V_x)\}_{x \in K}$ has a finite subcover. Let that be $\{\Gamma^{-1}(V_{x_1}), \cdots, \Gamma^{-1}(V_{x_k})\}$. Now we'll show that the sets $U_{x_i}^j$ for $i \in \{1, \ldots, k\}$ are a finite (sub)cover of $\Gamma(K)$. Let $z \in \Gamma(K)$. Then there exists some $\tilde{x} \in K$ such that $z \in \Gamma(\tilde{x})$. But $\{\Gamma^{-1}(V_{x_1}), \cdots, \Gamma^{-1}(V_{x_k})\}$ is a cover of K so by the definition of Γ^{-1} there exists some x_i such that $\Gamma(\tilde{x}) \subseteq V_{x_i}$. Since $z \in \Gamma(\tilde{x}) \subseteq V_{x_i} = U_{x_1}^1 \cup \cdots \cup U_{x_i}^{n_{x_1}}$, $z \in U_{x_i}^j$ for some j. This proves that the collection of U_x^j sets for $i \in \{1, \ldots, k\}$ is indeed a cover of $\Gamma(K)$.

6. Prove that if the graph of a correspondence is open, then the correspondence is lower hemicontinuous.

   Solution: Assume that the correspondence $\Gamma : X \to 2^Y$ (with $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$) has an open graph. Let $V \subseteq X$ be an open set such that for some $x \in X$ we have $\Gamma(x) \cap V \neq \varnothing$. Specifically, let $y \in \Gamma(x) \cap V$. Notice that $(x, y) \in \operatorname{graph} \Gamma$ and, since $\operatorname{graph} \Gamma$ is open, there exists some $\varepsilon > 0$ such that $B_\varepsilon((x, y)) \subseteq \operatorname{graph} \Gamma$.

   In the next step, we show that $B_\varepsilon(x) \times \{y\} \subseteq B_\varepsilon((x, y))$.[3] Indeed let

[3] Note that $B_\varepsilon(x) \subseteq X$, while $B_\varepsilon((x, y)) \subseteq X \times Y$.

$x' \in B_\varepsilon(x)$. Then

$$\begin{aligned}
\|(x', y) - (x, y)\| &= \sqrt{(x_1' - x_1)^2 + \cdots + (x_n' - x_n)^2 + (y_1 - y_1)^2 + \cdots + (y_m - y_m)^2} \\
&= \sqrt{(x_1' - x_1)^2 + \cdots + (x_n' - x_n)^2} \\
&= \|x' - x\| \\
&< \varepsilon
\end{aligned}$$

Hence $(x', y) \in B_\varepsilon(x, y)$ and $B_\varepsilon(x) \times \{y\} \subseteq B_\varepsilon((x, y)) \subseteq \operatorname{graph} \Gamma$. So $y \in \Gamma(x')$ for all $x' \in B_\varepsilon(x)$. Since $y \in \Gamma(x) \cap V \subseteq V$ we have $y \in \Gamma(x') \cap V \neq \varnothing$ for all $x' \in B_\varepsilon(x)$, proving that Γ is indeed lhc.