Problem 1.

Call a metric space (X, d) *discrete* if every subset $A \subset X$ is open. Prove or provide a counterexample: every discrete metric space is complete.

Solution

Counterexample: consider the metric space $X = \{\frac{1}{n} : n \in \mathbb{N}\}$ with the usual absolute value metric. First, for every $x = \frac{1}{n} \in X$, if $\varepsilon < \frac{1}{2n(n+1)}$ then $B_\varepsilon(x) = \{x\}$. Hence every singleton is open, and since every subset of X can be written as the union of singletons, every subset of X is open. So X is discrete. Further, the sequence $a_n = \frac{1}{n}$ is Cauchy in this metric space but is not convergent.

Remark: Using the same ambient space but the discrete metric induces the same *topology* (collection of open sets), but under the discrete metric the sequence defined above is not Cauchy!

Problem 2.

A function $f : X \to Y$ is *open* if for every open set $A \subset X$, its image $f(A)$ is also open. Show that any continuous open function from $\mathbb{R}$ into $\mathbb{R}$ (with the usual metric) is strictly monotonic.

Solution

A couple of points to note: first, for any $a < b \in \mathbb{R}$, compactness of $[a, b]$ and continuity of f gives us that $f([a, b])$ is compact. Denote the supremum and infimum as $M = \sup f([a, b])$ and $m = \inf f([a, b])$. The extreme value theorem gives us that $M, m \in f([a, b])$, i.e. we can find $p, q \in [a, b]$ such that $f(p) = M$ and $f(q) = n$.

Now, suppose the open mapping f is not strictly monotonic. So for some $a < c < b \in \mathbb{R}$, we have either (i) $f(a) \geq f(c) \leq f(b)$, or (ii) $f(a) \leq f(c) \geq f(b)$. In case (i), if $f(a) = M$ or $f(b) = M$, then $f(c) = M$. So $\sup f\left((a, b)\right) = M$ as well. But then $f\left((a, b)\right)$ is not open, since no open set of the entire real line can contain its own supremum. This is because every $B_\varepsilon(x)$ in the real number line contains elements both greater and less than x. This contradicts our assumption that f was an open mapping. Case (ii) is analogous. Hence f is strictly monotonic.

Problem 3.

Suppose f, g are continuous functions from metric spaces (X, d) into (Y, ρ). Let E be a dense subset of X (in a metric space, a set A is dense in B if $\overline{A} \supset B$, see correction!). Show that $f(E)$ is dense in $f(X)$. Further, if $f(x) = g(x)$ for every $x \in E$, then $f(x) = g(x)$ for every $x \in X$.

Solution

To show $f(E)$ is dense in $f(X)$, we need to show for every $y \in f(X)$, either $y \in f(E)$ or y is a limit point of E. So choose some $x \in X$ such that $f(x) = y$. Either $x \in E$ (in which case $f(x) = y \in f(E)$) or $x \in \overline{E} \setminus E$. In the latter case there exists a sequence $\{x_n\} \subset E$ such that $x_n \to x$. $x_n \in E \implies f(x_n) \in f(E)$ and continuity of f implies $f(x_n) \to f(x) = y$. Hence y is a limit point of $f(E)$.

Now suppose $f(x) = g(x)$ for every $x \in E$. Choose $x' \in X \setminus E$ and any sequence $\{x_n\} \subset E$ such that $x_n \to x'$. Then continuity guarantees that $f(x_n) \to f(x')$ and $g(x_n) \to g(x')$. But since $g(x_n) = f(x_n)$ for every $n \in \mathbb{N}$, the limit must be the same. So $f(x') = g(x')$.

Remark: This says that a continuous function is entirely determined by its values on any dense subset of its domain.

Correction: I had originally written "in a metric space, E is dense in X if $\overline{E} = X$." While this is true when X is the ambient metric space, in general a set A is dense in a set B if every element of B is either an element of A or a limit point of A. As written then, the problem is false. A student gave me the following counterexample: let $X = [0, 1] \cap \mathbb{Q}$, $Y = [0, 1]$, $E = [0, 1] \cap \mathbb{Q}$, and let $f : X \to Y$ be the identity function. f is continuous on X, E is dense in X, and note $f(E) = f(X) = [0, 1] \cap \mathbb{Q}$. But $\overline{f(E)} = [0, 1] \supsetneq f(X)$.

Problem 4.

Let (X, d) be a metric space.

(a) Suppose that for some $\varepsilon > 0$, every ε-ball in X has compact closure. Show that X is complete.

(b) Suppose that for each $x \in X$ there is an $\varepsilon > 0$ such that $B_\varepsilon(x)$ has compact closure. Show by means of an example that X need not be complete.

Solution

(a) Let $\varepsilon > 0$ be such that every ε-ball in X has compact closure and let $\{x_n\}$ be any Cauchy sequence in X. We know that there exists some N such that for all $m, n > N$ we have $d(x_n, x_m) < \varepsilon$. If we fix some $m > N$, for every $n > N$ this says that $x_n \in B_\varepsilon(x_m) \subset \overline{B_\varepsilon(x_m)}$.[1] The subsequence of $\{x_n\}_{n>N}$ is itself clearly a Cauchy sequence and is contained entirely in $\overline{B_\varepsilon(x_m)}$. So by sequential compactness, we can find a subsequence $\{x_{n_k}\}$ of $\{x_n\}_{n>N}$ such that $x_{n_k} \to x \in \overline{B_\varepsilon(x_m)}$. Recall that any Cauchy sequence with a convergent subsequence also converges to the same limit, so we have $x_n \to x \in \overline{B_\varepsilon(x_m)} \subset X$. Thus we have shown every Cauchy sequence has a limit contained in X.

(b) Let $X = (0, \infty)$ with the standard metric. Then for every $x \in X$, choose $\varepsilon = \frac{x}{2}$. Then $\overline{B_\varepsilon(x)} = \overline{\left(\frac{x}{2}, \frac{3x}{2}\right)} = \left[\frac{x}{2}, \frac{3x}{2}\right]$, which is a closed and bounded subset of the (strictly positive) reals and is therefore compact. However, this space is not a complete metric space because the Cauchy sequence $x_n = \frac{1}{n}$ does not converge.

[1] Despite appearance, the closure of an open ε-ball need not be the corresponding closed ball. Try to think of an example.

Problem 5.

Let (X, d) be a compact metric space and let $\Phi(x) : X \to 2^X$ be a upper-hemicontinuous, compact-valued correspondence, such that $\Phi(x)$ is non-empty for every $x \in X$. Prove that there exists a compact non-empty subset K of X, such that $\Phi(K) \equiv \bigcup_{x \in K} \Phi(x) = K$.

Solution

There's a lot to show in this one. Let's start here:

Lemma. *Let (X, d) be a metric space and let $\Psi(x) : X \to 2^X$ be a upper-hemicontinuous, compact-valued and non-empty correspondence. If $K \subset X$ is compact, then $\Psi(K)$ is compact.*

Proof. We will use the sequential characterization of upper-hemicontinuity and compactness. Choose any sequence $\{y_n\} \subset \Psi(K)$. So for every y_n we can find some x_n such that $y_n \in \Psi(x_n)$. Compactness of K means we can find a convergent subsequence $x_{n_k} \to x_0 \in K$. Then consider the corresponding subsequence $\{y_{n_k}\}$. By the sequential characterization of compact-valued and upper-hemicontinuous correspondences we can find a convergent (sub)subsequence $y_{n_{k_j}} \to y_0 \in \Psi(x_0)$. But this (sub)subsequence is itself a subsequence of $\{y_n\}$, and $x_0 \in K \implies \Psi(x_0) \subset \Psi(K)$. Hence for an arbitrary sequence in $\Psi(K)$ we can find a convergent subsequence whose limit lies in $\Psi(K)$. Thus the set is sequentially compact, hence compact. $\square$

Also, note that $A \subset B \implies \Psi(A) = \bigcup_{a \in A} \Psi(a) \subset \bigcup_{b \in B} \Psi(b) = \Psi(B)$ for any correspondence Ψ. So let's construct the following sequence of sets:

$$
\begin{aligned}
K_0 &= X \\
K_1 &= \Phi(K_0) \\
&\vdots \\
K_n &= \Phi(K_{n-1}) \\
&\vdots
\end{aligned}
$$

Using our Lemma, we can see inductively that that $K_0, K_1, \ldots$ are a sequence of nested, non-empty and compact sets. Then Cantor's intersection theorem tells us that $K = \bigcap_{n=0}^{\infty} K_n$ is non-empty. Since K is the intersection of closed sets, it is also closed. Then K is a closed subset of a compact metric space, so it is also compact.[2] Now I claim that $K = \Phi(K)$ otherwise why would I be doing all this?

First the easy direction: since $K \subset K_n$ for all n, we have $\Phi(K) \subset \Phi(K_n) = K_{n+1}$. Thus $\Phi(K) \subset K$. The other direction is more difficult, and the notation gets a bit cumbersome.

To show $K \subset \Phi(K)$, choose any $y_0 \in K$. Note for every n, we have $y_0 \in K_{n+1} = \Phi(K_n)$, so let's construct a sequence $\{x_n\}$ such that $x_n \in K_n$ and $y_0 \in \Phi(x_n)$. Since $\{x_n\} \subset K_0$, by compactness we can find a convergent subsequence $\{x_{n_j}\}$ with limit x_0. From how we have constructed the sequence, $\{x_n\}_{n \geq N}$ is entirely contained in K_N. But then for every N we

[2] In fact any closed subset of a compact *set* is compact.

can find some J such that $\{x_{n_j}\}_{j \geq J}$ is entirely contained in K_N. Hence x_0 is a limit point of every $K_N \implies x_0 \in K_N \; \forall N \implies x_0 \in K$.

Now finally, we have $y_0 \in \Phi(x_{n_j})$ for every n_j. Then this defines a constant sequence $y_{n_j} = y_0$, which of course converges to y_0 (along with all its subsequences). Using the sequential characterization of upper-hemicontinuous compact-valued correspondences, we know that $y_0 \in \Phi(x_0)$. Since we showed that $x_0 \in K$, we have $y_0 \in \Phi(K)$. y_0 was an arbitrary element of K, we have $K \subset \Phi(K)$.

Problem 6.

Define the correspondence $\Gamma : [0,1] \to 2^{[0,1]}$ by:

$$\Gamma(x) = \begin{cases} [0,1] \cap \mathbb{Q} & \text{if } x \in [0,1] \backslash \mathbb{Q} \\ [0,1] \backslash \mathbb{Q} & \text{if } x \in [0,1] \cap \mathbb{Q} \end{cases}.$$

Show that Γ is not continuous, but it is lower-hemicontinuous. Is Γ upper-hemicontinuous at any rational? At any irrational? Does this correspondence have a closed graph?

Solution

Consider the open set $V = (0,1)$ which contains $\Gamma(q) = [0,1] \backslash \mathbb{Q}$ for every $q \in [0,1] \cap \mathbb{Q}$. Then any open set containing q will also contain an irrational number $x \in [0,1] \setminus \mathbb{Q}$, and $\Gamma(x) = [0,1] \cap \mathbb{Q} \not\subset V$. Hence Γ is not upper-hemicontinuous at any rational number.

Now fix some $y \in [0,1] \setminus \mathbb{Q}$ and consider the open set $V = (-1,y) \cup (y,2)$. For any $x \in [0,1] \setminus \mathbb{Q}$ we have $\Gamma(x) \subset V$, but every open set containing x will also contain a rational number $q \in [0,1] \cap \mathbb{Q}$ and $\Gamma(q) = [0,1] \setminus \mathbb{Q} \not\subset V$. Thus Γ is nowhere upper-hemicontinuous and hence nowhere continuous.

Next, let V be any open set satisfying $V \cap [0,1] \neq \varnothing$. Then we have $V \cap ([0,1] \cap \mathbb{Q}) \neq \varnothing$ and $V \cap ([0,1] \setminus \mathbb{Q}) \neq \varnothing$, since every ε-ball in the reals contains both rational and irrational numbers. But then $\Gamma(x) \cap V \neq \varnothing$ for every x in the domain of Γ. This proves that Γ is lower-hemicontinuous.

The correspondence does not have a closed graph. Remember that $\mathrm{gr}(\Gamma)$ is a subset of $[0,1] \times [0,1]$. Fix some $y \in [0,1] \setminus \mathbb{Q}$ and take any sequence $\{q_n\} \subset [0,1] \cap \mathbb{Q}$ such that $q_n \to y$. Then the sequence $(q_n, y) \in \mathrm{gr}(\Gamma)$ but $(y,y) \notin \mathrm{gr}(\Gamma)$. Hence the graph is not closed.