Economics 204
Fall 2011
Problem Set 4
Due Tuesday, August 7 in Lecture

1. Consider $\mathcal{C}^n([0,1])$, the vector space of real-valued n-times differentiable functions with a continuous n-th derivative on the unit interval, and equip it with a supremum norm[1]
$$\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|$$
   (a) Prove that it is a normed vector space.[2]
   (b) Define $T_n : \mathcal{C}^n([0,1]) \to \mathcal{C}^{n-1}([0,1])$ by $T_n(f) = \frac{d^n}{dx^n}(f)$, i.e. the n-th derivative of f. Prove that T_n is a linear mapping.
   (c) Find the dimension and provide a basis for $\ker T_n$.
   (d) Define $S_1 : \mathcal{C}([0,1]) \to \mathcal{C}^1([0,1])$ by $S_1(f)(x) = \int_0^x f(t)\,dt$. Show that $T_1 S_1$ is the identity map on $\mathcal{C}([0,1])$ despite the noninvertibility of T_1.

2. Show that for any subset U of the vector space, the span of the span equals to the span
$$\mathrm{span}\,(S) = \mathrm{span}\,(\,\mathrm{span}\,(S)).$$

3. Let X be a vector space. Let $T : X \to X$ and $U : X \to X$ be linear transformations such that $\ker T$ and $\ker U$ are finite-dimensional and U is surjective, that is, $U(X) = X$.
   (a) Verify directly that $\ker\,(T \circ U)$ is a vector subspace of X.
   (b) Show that $\ker\,(T \circ U)$ is finite dimensional and that
$$\dim\ker(T \circ U) = \dim\ker\,T + \dim\ker\,U.$$

4. Let X and Y be finite-dimensional linear spaces with $\dim X = n$ and $\dim Y = m$. Let $T : X \to Y$ be a linear transformation. Show that there are bases V and W such $Mtx_{W,V}(T)$ is upper-triangular (that is, all elements with $i > j$ are zeros).

[1] By convention we set $\mathcal{C}^0([0,1]) = \mathcal{C}([0,1])$, the space of all continuous functions on unit interval.
[2] You may take it as given that the set of all real-valued functions defined on $[a,\,b]$ is a vector space over $\mathbf{R}$.

5. Let X be a finite-dimensional vector space with a basis V, and let $T : X \to X$ be linear transformation.

   (a) Show that T is invertible if and only if $Mtx_V(T)$ is invertible. (Hint: Use the commutative diagram.

   (b) Show that $Mtx_V(T^{-1}) = (Mtx_V(T))^{-1}$.

6. Let A, B be $n \times n$ matrices. Prove that AB has the same eigenvalues as BA.

7. Similarity

   (a) Does similarity constitutes an equivalence relation among square matrices?

   (b) Show that similar matrices have the same determinant.

   (c) Show that if A is similar to B and A is nonsingular then B is nonsingular and A^{-1} is similar to B^{-1}.

   (d) Show that if A and B are similar and λ is a scalar then $A - \lambda I$ and $B - \lambda I$ are similar.