Economics 204
Fall 2013
Problem Set 4
Due Tuesday, August 13 in Lecture

1. Suppose that V is finite dimensional and U is a subspace of V such that $\dim U = \dim V$. Prove that $U = V$.

2. $T : M_{2\times 3} \to M_{2\times 2}$ is defined by

$$T\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} 2a_{11} - a_{12} & a_{13} + 2a_{12} \\ 0 & 0 \end{pmatrix}$$

Determine $\mathrm{Ker}\,(T)$, $\dim \mathrm{Ker}\,(T)$ and $\mathrm{Rank}\,(T)$. Is T one-to-one, onto, or neither?

3. Prove the Theorem 1 in Lecture 9, that is, if $\dim X < \infty$ then for any subspace W of X we must have

$$\dim W + \dim(X/W) = \dim X$$

by, first, showing that a basis $\{w_1, w_2, \ldots, w_m\}$ of W can be extended to the basis $\{w_1, w_2, \ldots, w_n\}$ of X, and then proving that $\{[w_{m+1}], [w_{m+2}], \ldots, [w_n]\}$ is the basis of X/W.

4. Derive a transformation, $T : \mathbf{R}^2 \to \mathbf{R}^2$, which reflects a point across the line $y = 5x$.

   (a) First, calculate the action of T on the points $(1, 5)$ and $(-5, 1)$.

   (b) Next, write the matrix representation of T using these two vectors as a basis.

   (c) Find S and S^{-1}, the matrices that change coordinates under this basis to standard coordinates and back again.

   (d) Write the matrix representation of T in the standard basis.

   (e) Use the point $(-5, 1)$ to verify the commutative diagram.

5. Prove the following useful facts about eigenvalues:

   (a) Eigenvalues of any upper or lower triangular matrix A are the diagonal entries of A.

   (b) Show that if λ is an eigenvalue of A then λ^k is an eigenvalue of A^k for $k \in \mathbf{N}$.

6. Let A be the $n \times n$ matrix which has zeros on the main diagonal and ones everywhere else. Find the eigenvalues and eigenspaces of A and compute $\det(A)$.

   Note that we define an eigenspace of matrix as the set of all its eigenvectors, together with the zero vector. Formally, an eigenspace of matrix A associated with an eigenvalue λ is $V_\lambda = \{\, x |\, Ax = \lambda x\} \cup \{0\}$.

7. The Supremum Norm on $L(\mathbf{R}^k,\, \mathbf{R}^k)$

   (a) Compute the norms of the following matrices:

   $$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix},\ B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},\ AB = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$$

   (b) Prove that for any $A, B \in L(\mathbf{R}^k,\, \mathbf{R}^k)$ we have the following inequality:

   $$\|AB\| \,\leq\, \|A\|\, \|B\|$$

   Show that *non-strict inequality* is really what we need by finding a pair of matrices such that the inequality is strict, and another pair of nonzero matrices such that equality holds.

   (c) Prove that the subset of $L(\mathbf{R}^k,\, \mathbf{R}^k)$ consisting of all invertible linear operators is *open* under the topology induced by the supremum norm. You may want to employ the following steps:

      i. Step 1: Let $A, B \in L(\mathbf{R}^k,\, \mathbf{R}^k)$. Show that if A is invertible and B satisfies:

      $$\|B - A\|\, \|A^{-1}\| < 1$$

      Then B is one-to-one.

      ii. Step 2: Show that if B is one-to-one, then B is onto (and hence invertible).

      iii. Step 3: Show that there is a ball with center at A comprised entirely of invertible operators.