Economics 204
Fall 2012
Problem Set 4 Suggested Solutions

1. Consider $\mathcal{C}^n([0,1])$, the vector space of real-valued $n$-times differentiable functions with a continuous $n$-th derivative on the unit interval, and equip it with a supremum norm[1]

$$\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|$$

   (a) Prove that it is a normed vector space.[2]

   **Solution.** First, we must show that $\mathcal{C}^n([0,1])$ is a vector space. As we already know that the set of all functions mapping $[a,b]$ to $\mathbf{R}$ is a vector space over $\mathbf{R}$, all we need to show is that $\mathcal{C}^n([0,1])$ is a vector subspace of this set. Note that the sum of real-valued $n$-times differentiable functions with a continuous $n$-th derivative is real-valued $n$-times differentiable function with a continuous $n$-th derivative. Also, note that a scalar multiple of a real-valued $n$-times differentiable functions with a continuous $n$-th derivative is real-valued $n$-times differentiable functions with a continuous $n$-th derivative $\mathcal{C}^n([0,1])$ is closed to vector addition and scalar multiplication, and is therefore a vector space.

   Next, we must show that $||f||_\infty$ satisfies the properties of a norm. First, $||f||_\infty$ is clearly $\geq 0$ $\forall f \in \mathcal{C}^n([0,1])$ as a consequence of the absolute value operator. Second, it is also clear that $||f||_\infty = 0 \iff f(x) = 0$ $\forall x \in [0,1]$. Third, we must show that $||f||_\infty$ satisfies the triangle inequality. Using $(f+g)(x) = f(x) + g(x)$, we have that

$$\sup_{x\in[0,1]} |f(x)+g(x)| \leq \sup_{x\in[0,1]} f(x) + \sup_{x\in[0,1]} g(x)$$

   because the right-hand side allows us to pick a different $x \in [0,1]$ for each part of the sum. Thus, the triangle inequality is satisfied, and $\mathcal{C}^n([0,1])$ equipped with $||f||_\infty$ is a normed vector space.

   (b) Define $T_n : \mathcal{C}^n([0,1]) \to \mathcal{C}([0,1])$ by $T_n(f) = \frac{d^n}{dx^n}(f)$, i.e. the $n$-th derivative of $f$. Prove that $T_n$ is a linear mapping.

   **Solution.** The statement of the problem has $T_n : \mathcal{C}^n([0,1]) \to \mathcal{C}^{n-1}([0,1])$ but it clearly should have been $T_n : \mathcal{C}^n([0,1]) \to \mathcal{C}([0,1])$ since $T_n$ is defined as $n$-times differentiation. Sorry about that.

   Now, the result follows immediately from the linearity of differentiation.

[1] By convention we set $\mathcal{C}^0([0,1]) = \mathcal{C}([0,1])$, the space of all continuous functions on unit interval.

[2] You may take it as given that the set of all real-valued functions defined on $[a, b]$ is a vector space over $\mathbf{R}$.

(c) Find the dimension and provide a basis for $\ker T_n$.

**Solution.** A continuously differentiable function vanishes after taking $n$ derivatives iff it is a polynomial of degree at most $n-1$ (this can be verified by taking the $n$-fold antiderivative of the 0 function and noting that antiderivatives are unique up to a constant). It follows that a basis for $\ker T_n$ is $\{1, x, x^2, \ldots, x^{n-1}\}$.

(d) Define $S_1 : \mathcal{C}([0,1]) \to \mathcal{C}^1([0,1])$ by $S_1(f)(x) = \int_0^x f(t)\,dt$. Show that $T_1 S_1$ is the identity map on $\mathcal{C}([0,1])$ despite the noninvertibility of $T_1$.

**Solution.** The Fundamental Theorem of Calculus states that if $f(x)$ is continuous on $[0,1]$, then the function $F$ defined by $F(0) = \alpha$, $F(x) - F(0) = \int_0^x f(t)\,dt$ is differentiable on $[0,1]$, with derivative $F'(x) = f(x)$. It follows that $T_1 S_1 f(x) = \frac{d}{dx}\left(\int_0^x f(t)dt\right) = f(x)$. $T_1 S_1$ is thus the identity linear map.

2. Show that for any subset $U$ of the vector space, the span of the span equals to the span

$$\operatorname{span}(S) = \operatorname{span}(\operatorname{span}(S)).$$

**Solution.** We show this by two-way set inclusion.

$\operatorname{span}(\operatorname{span}(S)) \supseteq \operatorname{span}(S)$ containment follows immediately because for any subset $A$ of some vector space, $\operatorname{span}(A) \subseteq A$.

$\operatorname{span}(\operatorname{span}(S)) \subseteq \operatorname{span}(S)$. Consider $m$ vectors from $\operatorname{span}(S)$, namely $c_1^1 s_1^1 + \cdots + c_{n_1}^1 s_{n_1}^1, \ \ldots, \ c_1^m s_1^m + \ldots + c_{n_m}^m s_{n_m}^m$ (where superscript refers to the $m$-th vector and subscript to $n$-th vector in the linear combination of $m$-th vector in the *span*), and note that any linear combination of those

$$\alpha_1(c_1^1 s_1^1 + \cdots + c_{n_1}^1 s_{n_1}^1) + \cdots + \alpha_m(c_1^m s_1^m + \ldots + c_{n_m}^m s_{n_m}^m)$$

is just a linear combination of elements in $S$

$$(\alpha_1 c_1^1)s_1^1 + (\alpha_1 c_{n_1}^1)s_{n_1}^1 + (\alpha_m c_1^m)s_1^m + (\alpha_m c_{n_m}^m)s_{n_m}^m$$

which means it is in the $\operatorname{span}(S)$.

3. Let $X$ be a vector space. Let $T : X \to X$ and $U : X \to X$ be linear transformations such that $\ker T$ and $\ker U$ are finite-dimensional and $U$ is surjective, that is, $U(X) = X$.

(a) Verify directly that $\ker(T \circ U)$ is a vector subspace of $X$.

**Solution.** We need to check that linear combination of elements from $\ker(T \circ U)$ belongs again to $\ker(T \circ U)$, but this is immediate given that composition of linear transformations $T \circ U$ is a linear transformation.

(b) Show that $\ker(T \circ U)$ is finite dimensional and that

$$\dim \ker(T \circ U) = \dim \ker T + \dim \ker U.$$

**Solution.** We know that both $\ker U$ and $\ker T$ are finite-dimensional, so let $\{v_1, v_2, \ldots, v_m\}$ and $(w_1, w_2, \ldots, w_n)$ be their bases.
Observe that by definition $\ker U \subseteq \ker(T \circ U)$. Thus, dimension of the $\ker(T \circ U)$ have to be at least $m$, and, at the same time, it can't be greater then $m+n$, as $T \circ U$ is just a composition of two linear transformations, each with finite-dimensional kernel. Lets show that it equals exactly to $m+n$, and to this end, lets prove a claim that $\{v_1, v_2, \ldots, v_m\}$ and $\{u_1, u_2, \ldots, u_n\}$ form basis for $\ker(T \circ U)$, where $Uu_i = w_i$ for all $i = 1, \ldots, n$. Such $u_i$'s clearly exist because $U$ is a surjective.

Firstly, note that $\{u_1, u_2, \ldots, u_n\}$ are clearly independent (follows immediately by applying $U$ to a linear combination of $u_i$s and recalling that $(w_1, w_2, \ldots, w_n)$ are linear independent. Lets show that $\{v_1, v_2, \ldots, v_m\}$ and $\{u_1, u_2, \ldots, u_n\}$ are linear independent collection of vectors in $X$. So, take non-zero scalars $\gamma_1, \gamma_2, \ldots, \gamma_n, \gamma_{n+1}, \ldots, \gamma_{m+n}$ and suppose that

$$\gamma_1 \cdot u_1 + \gamma_2 \cdot u_2 + \cdots + \gamma_n \cdot u_n + \gamma_{n+1} \cdot v_1 + \cdots + \gamma_{m+n} \cdot v_m = 0.$$

Applying $U$ to this linear combination implies that $\gamma_i = 0$ for all $i = 1, \ldots, n$ because $\{u_1, u_2, \ldots, u_n\}$ are linearly independent. Also, $\gamma_i = 0$ for all $i = n+1, \ldots, n+m$ is also immediate since $\{v_1, v_2, \ldots, v_m\}$ are the basis and we get the result we desire.

4. Let $X$ and $Y$ be finite-dimensional linear spaces with $\dim X = n$ and $\dim Y = m$. Let $T : X \to Y$ be a linear transformation. Show that there are bases $V$ and $W$ such $Mtx_{W,V}(T)$ is upper-triangular (that is, all elements with $i > j$ are zeros).

**Solution.** We will prove this result, also known as "Schur's triangulation," in following three steps.

**Step 1.** Let $Z$ be a subspace of $X$ and $V_Z = (v_1, \ldots, v_k)$ be a base for $Z$. Because $Z$ is a subspace we can extend $V_Z$ to $V$, where $V$ is a basis of $X$, such that $V_Z \subseteq V$, i.e. $V = (v_1, \ldots, v_k, v_{k+1}, \ldots, v_n)$.

Now, let $x$ be an arbitrary vector in $Z$ then

$$crd_{V_Z}(x) = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_k \end{pmatrix} \in \mathbf{R^k}$$

but when $x$ viewed as an element of $X$ it has following representation

$$crd_V(x) = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_k \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbf{R^n}.$$

**Step 2.** Lets consider a linear transformation $T : X \to Y$ along with its restriction $T|_Z : Z \to Y$. The rangespace of $T|_Z$ is a subspace of $Y$, so lets fix a basis $W_{T|_Z}$, which for ease of notation we will denote by $U$ and then extend it to a basis $W$ for $Y$.

Now, lets observe that

$$Mtx_{U,V_Z}(T|_Z) = \begin{pmatrix} \alpha_{1,1} & \dots & \alpha_{1,n} \\ \vdots & \ddots & \vdots \\ \alpha_{r,1} & \dots & \alpha_{r,n} \end{pmatrix}$$

and

$$Mtx_{W,V}(T) = \begin{pmatrix} \alpha_{1,1} & \dots & \alpha_{1,k} & \alpha_{1,k+1} & \dots & \alpha_{1,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{r,1} & \dots & \alpha_{r,k} & \alpha_{r,k+1} & \dots & \alpha_{r,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \alpha_{m,k+1} & \dots & \alpha_{m,n} \end{pmatrix}$$

because of the reasoning given in step 1.

**Step 3.** Lets $V = (v_1, \dots, v_n)$ be a basis for $X$. Observe that the spans

$$\text{span}(\{0\}) = \{0\} \subset \text{span}(\{v_1\}) \subset \text{span}(\{v_1, v_2\}) \subset \dots \subset \text{span}(V) = V$$

form a strictly increasing chain of subsets.

Now, take any linear map $T : X \to Y$ and note that if we take

$$Y_i = \text{span}(\{T(v_1), \dots, T(v_i)\})$$

then $Y_i$ will form an increasing chain of subsets of $Y$

$$Y_0 = \{0\} \subseteq Y_1 \subseteq \dots \subseteq Y_m = Y$$

with a property that $T(\text{span}(\{v_1, \dots, v_i\})) \subset Y_i$ for each $i$ and the desired result follows immediately.

5. Let $X$ be a finite-dimensional vector space with a basis $V$, and let $T : X \to X$ be linear transformation.

   (a) Show that $T$ is invertible if and only if $Mtx_V(T)$ is invertible. (Hint: Use the commutative diagram.)

   **Solution.**
   ($\Longrightarrow$) Since $T$ is invertible $TT^{-1}(x) = x$ for all $x \in X$. Using matrix representation of $x$ with respect to $V$ we get

$$\begin{aligned} crd_V(TT^{-1}(x)) &= crd_V(x) \implies \\ Mtx_V(T)Mtx_V(T^{-1})crd_V(x) &= crd_V(x) \end{aligned}$$

   for all vectors of the form $crd_V(x)$.
   Similarly, $T^{-1}T(x) = x$ implies

$$Mtx_V(T^{-1})Mtx_V(T)crd_V(x) = crd_V(x).$$

   Therefore, $Mtx_V(T)$ is invertible.

   ($\Longleftarrow$) Since $Mtx_V(T)$ is invertible lets define

$$S(x) = crd_V^{-1} \circ Mtx_V(T)^{-1} \circ crd_V(x),$$

   which is clearly invertible. By the commutative diagram,

$$ST(x) = crd_V^{-1} \circ Mtx_V(T)^{-1} \circ crd_V(T(x)) = crd_V^{-1} \circ crd_V(x) = x$$

   and

$$TS(x) = T(crd_V^{-1} \circ Mtx_V(T)^{-1} \circ crd_V(x)) = x$$

   for every $x \in X$. Therefore, $TS = ST = id$ and $T$ is invertible.

   (b) Show that $Mtx_V(T^{-1}) = (Mtx_V(T))^{-1}$.

   **Solution.** If $T$ invertible $TT^{-1}(x) = x$ for all $x \in X$. This implies directly

$$Mtx_V(T)Mtx_V(T^{-1}) = id.$$

   Since $Mtx_V(T)$ is invertible we get $Mtx_V(T^{-1}) = (Mtx_V(T))^{-1}$.

6. Let $A$, $B$ be $n \times n$ matrices. Prove that $AB$ has the same eigenvalues as $BA$.

   **Solution.** Let $\lambda$ is an eigenvalue of $AB$, which means $ABx = \lambda x$ for some non-zero vector $x$. Let multiply both sides of inequality by $B$ and consider $y = Bx$. There are two possible cases.

   Case $y = 0$. Then $Ay = ABx = \lambda x = 0$, and since $x$ was non-zero by assumption this means $\lambda = 0$. Thus we have

$$\begin{aligned} \det(BA - \lambda I) = \det(BA) = \det(B)\det(A) = \\ = \det(A)\det(B) = \det(AB) = \det(AB - \lambda I) = 0 \end{aligned}$$

and $\lambda$ is an eigenvalue of $BA$.

Case $y \neq 0$. Then $\lambda$ is clearly an eigenvalue of $BA$ with $y$ as an eigenvector and we are done.

7. Similarity

(a) Does similarity constitutes an equivalence relation among square matrices?

**Solution.** Yes, it is an equivalence relation. First, $I = I^{-1}$ and for every $n \times n$ matrix $A$, $A = AI = IAI = I^{-1}AI$, so $A$ is similar to itself. Second, if $A = P^{-1}BP$, then $PAP^{-1} = PP^{-1}BPP^{-1} = B$, so if $A$ is similar to $B$ then $B$ is similar to $A$.

Finally, suppose matrices $A$ and $B$ are similar, and that matrices $B$ and $C$ are similar. We must show that $A$ and $C$ are similar. Let $B = P^{-1}AP$, and let $C = Q^{-1}BQ$. Substituting, we have $C = Q^{-1}P^{-1}APQ$. Define $R = PQ$. By Theorem 4.12 in de la Fuente, $R$ is invertible and $R^{-1} = Q^{-1}P^{-1}$. Thus, $C = R^{-1}AR$, and $A$ is similar to $C$.

(b) Show that similar matrices have the same determinant.

**Solution.** The key to this proof is the fact that the determinant of the product of any matrices is equal to the product of the matrices' determinants. Start by letting $A$ be similar to $B$, so that $A = P^{-1}BP$. Then $|A| = |P^{-1}||B||P|$. Then, because scalar multiplication is commutative, we have $|A| = |B||P^{-1}||P| = |B|$.

(c) Show that if $A$ is similar to $B$ and $A$ is nonsingular then $B$ is nonsingular and $A^{-1}$ is similar to $B^{-1}$.

**Solution.** $A$ is similar to $B$, so let $B = P^{-1}AP$. Post-multiplying both sides of this equality by $P^{-1}A^{-1}P$ (which we can do since both $A$ and $P$ are invertible), we find

$$B(P^{-1}A^{-1}P) = P^{-1}A^{-1}PP^{-1}AP = P^{-1}A^{-1}AP = P^{-1}P = I.$$

We have therefore found a matrix $Z = P^{-1}A^{-1}P$ such that $BZ = I$. Therefore, $B$ is invertible and non-singular, and $B^{-1} = Z = P^{-1}A^{-1}P$, so that $B^{-1}$ is similar to $A^{-1}$.

(d) Show that if $A$ and $B$ are similar and $\lambda$ is a scalar then $A - \lambda I$ and $B - \lambda I$ are similar.

**Solution.** Let $A = P^{-1}BP$. Then $A - \lambda I = P^{-1}BP - \lambda I = P^{-1}BP - \lambda P^{-1}IP$, where the last equality follows from $P^{-1}IP = I$. Applying the distributive property of matrix multiplication, we therefore find that $A - \lambda I = P^{-1}(B - \lambda I)P$, so that $A - \lambda I$ and $B - \lambda I$ are similar.