Economics 204
Fall 2013
Problem Set 4 Suggested Solutions

1. Suppose that V is finite dimensional and U is a subspace of V such that $\dim U = \dim V$. Prove that $U = V$.

   Solution. Let $\{u_1, \ldots, u_n\}$ be a basis of U. Thus, $n = \dim U$, and by hypothesis we also have $n = \dim V$. Therefore, $\{u_1, \ldots, u_n\}$ is linear independent collection of vectors in V. Theorem 9 in Lecture Notes 8 on page 4 states that $\{u_1, \ldots, u_n\}$ is a basis of V. In particular, every vector in V is a linear combination of $\{u_1, \ldots, u_n\}$. Because for all i we have $u_i \in U$, it must be the case that $U = V$. We are done.

2. $T : M_{2\times 3} \to M_{2\times 2}$ is defined by

$$T\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} 2a_{11} - a_{12} & a_{13} + 2a_{12} \\ 0 & 0 \end{pmatrix}$$

   Determine $\ker(T)$, $\dim \ker(T)$ and $\operatorname{rank}(T)$. Is T one-to-one, onto, or neither?

   Solution. We start by noting that

$$\begin{aligned} \ker(T) &= \{m \in M_{3\times 2} : 0\} \\ &= \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} : a_{12} = 2a_{11},\, a_{13} = -4a_{11} \right\}. \end{aligned}$$

   It is easy to see that $\ker T$ is a linear subspace and its dimension is 4, an example of the basis of $\ker T$ for instance is a following set of vectors

$$\left\{ \begin{pmatrix} 1 & 2 & -4 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\},$$

   where linear independence of those vectors and spanning property follow immediately. By the Rank-Nullity Theorem $\operatorname{Rank}(T) = 2$. Note, since $\ker(T) \neq \{\vec{0}\}$ T is not one-to-one, or, alternatively, note that the basis for $\ker(T)$ given above comprises of all distinct vectors but they all map into $\vec{0}$.

   Since $\operatorname{Rank}(T) = 2 < \dim(M_{2\times 2})$ T is not onto. Alternatively, there are no $m \in M_{2\times 2}$ of the form $\begin{pmatrix} 0 & 0 \\ r & 0 \end{pmatrix}$ for some $r \neq 0$.

3. Prove the Theorem 1 in Lecture 9, that is, if $\dim X < \infty$ then for any subspace W of X we must have

$$\dim W + \dim(X \diagup W) = \dim X$$

by, first, showing that a basis $\{w_1, w_2, \ldots, w_m\}$ of W can be extended to the basis $\{w_1, w_2, \ldots, w_n\}$ of X, and then proving that $\{[w_{m+1}], [w_{m+2}], \ldots, [w_n]\}$ is the basis of $X \diagup W$.

Solution. We will follow suggestion given in the problem.

Step 1. Given a basis $\{w_1, ..., w_m\}$ of W, lets show that it can be extended it to a basis $\{w_1, ..., w_n\}$ of X. So, let $\dim W = m$, $\dim X = n$. If $m = n$ then we're done, since then the linearly independent set of vectors $\{w_1, ..., w_m\}$ will be a basis for X.

Now, if $m < n$ then $\{w_1, ..., w_m\}$ is not a basis of X, and hence $\text{span}\{w_1, ..., w_m\}$ is not all of X. That is, there exists a vector in X but not $\text{span}\{w_1, ..., w_m\}$; call it w_{m+1}. We claim that $\{w_1, ..., w_{m+1}\}$ is still linearly independent; the proof (by contradiction) is short and is left as an exercise. Now if $m + 1 = n$ then we are done, but if $m + 1 < n$ then repeat this same argument and add a w_{m+2} to the set. We can repeat this argument at most $n - m$ times and the basis of X that we wind up with will have $\{w_1, ..., w_m\}$ as its first m vectors.

Step 2. Lets show that $\{[w_{m+1}], ..., [w_n]\}$ is a basis of X/W, by showing first that its a linearly independent set. Let $\alpha_{m+1}, \ldots, \alpha_n$ be scalars such that $\sum_{j=m+1}^{n} \alpha_j [w_j] = [0]$; $[0]$ is the zero vector in X/W, and is, in fact, equal to W. Now $\sum_{j=m+1}^{n} \alpha_j [w_j] = \left[\sum_{j=m+1}^{n} \alpha_j w_j\right] = [0]$ implies that $\sum_{j=m+1}^{n} \alpha_j w_j \in W$, but unless each $\alpha_j = 0$ this is a contradiction since $w_{m+1}, \ldots, w_n$ are by construction elements not in the span of a basis of W, and hence neither of linear combinations of $w_{m+1}, \ldots, w_n$. Therefore, $\{[w_{m+1}], \ldots, [w_n]\}$ is linearly independent.

Next, lets show that it spans X/W. Let $[x] \in X/W$. We have two cases to consider. First, if $x \in W$ then we have $0[w_{m+1}] + \cdots + 0[w_n] = [0] = [x]$. Second, if $x \notin W$, then note that we can certainly write $x = \sum_{j=1}^{n} \alpha_j w_j$ for some $\alpha_1, \ldots, \alpha_n$. Now $\sum_{j=1}^{m} \alpha_j w_j \in W$, so $\sum_{j=m+1}^{n} \alpha_j w_j \in [x]$, and as you should verify, $a \in [b] \Leftrightarrow [a] = [b]$. Therefore $\left[\sum_{j=m+1}^{n} \alpha_j w_j\right] = [x]$, yielding $\sum_{j=m+1}^{n} \alpha_j [w_j] = [x]$ as desired and we are done.

4. Derive a transformation, $T : \mathbf{R}^2 \to \mathbf{R}^2$, which reflects a point across the line $y = 5x$.

   (a) First, calculate the action of T on the points $(1, 5)$ and $(-5, 1)$.

   Solution. Since $(1, 5)$ lies on the line $y = 5x$ it is unchanged by T so we know that $T(1, 5) = (1, 5)$. The slope of the line $y = 5x$ is 5 and the slope of the vector $(-5, 1)$ is $-1/5$. Because the vector is perpendicular with the line, reflecting it across the line takes is to $(5, -1)$, so $T(-5, 1) = (5, -1)$.

(b) Next, write the matrix representation of T using these two vectors as a basis.

Solution. Let

$$V = \{v_1, v_2\} = \left\{ \begin{pmatrix} 1 \\ 5 \end{pmatrix}, \begin{pmatrix} -5 \\ 1 \end{pmatrix} \right\}.$$

From the first question we know that $T(v_1) = v_1 = 1 \cdot v_1 + 0 \cdot v_2$ and $T(v_2) = -v_2 = 0 \cdot v_1 - 1 \cdot v_2$. So we write

$$P = Mtx_V(T) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(c) Find S and S^{-1}, the matrices that change coordinates under this basis to standard coordinates and back again.

Solution. We can easily find S, the matrix that changes coordinates in V to coordinates under the standard basis, E, because we already expressed the basis vectors v_1 and v_2 in terms of standard basis coordinates. We have

$$S = Mtx_{E,V}(id) = \begin{pmatrix} v_1 & v_2 \end{pmatrix} = \begin{pmatrix} 1 & -5 \\ 5 & 1 \end{pmatrix}.$$

The matrix that changes coordinates from E to V is simply the inverse of S, thus

$$S^{-1} = Mtx_{V,E}(id) = \frac{1}{26} \begin{pmatrix} 1 & 5 \\ -5 & 1 \end{pmatrix} = \begin{pmatrix} 1/26 & 5/26 \\ -5/26 & 1/26 \end{pmatrix}.$$

(d) Write the matrix representation of T in the standard basis.

Solution. One way to find $Mtx_E(T)$ would be to calculate $T(e_1)$ and $T(e_2)$ put the coordinates of these vectors in the columns of $Mtx_E(T)$. However, since we are not given a formula for T—just a description of its action—it takes a little work to find $T(e_1)$ and $T(e_2)$. (The reason we use V as a basis is because it the action of T on these basis vectors is straightforward.) Instead, we will apply the commutative diagram by changing coordinates to V, applying T, and changing back to E. In other words,

$$Mtx_E(T) = Mtx_{E,V}(id) \cdot Mtx_V(T) \cdot Mtx_{V,E}(id) = SPS^{-1}$$

and this equals

$$\begin{pmatrix} 1 & -5 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1/26 & 5/26 \\ -5/26 & 1/26 \end{pmatrix} = \begin{pmatrix} -24/26 & 10/26 \\ 10/26 & 24/26 \end{pmatrix}.$$

(e) Use the point $(-5, 1)$ to verify the commutative diagram.

Solution. We established in (a) that $T(-5, 1) = (5, -1)$ so now we will verify that multiplying $SPS^{-1}(-5, 1)^\top$ yields the same result. Writing out this computation, we have

$$\begin{aligned} SPS^{-1}(-5,1)^\top &= (SP)(S^{-1}(-5,1)^\top) = \\ &= (SP)(0,1)^\top = S(P(0,1)^\top) = S(0,-1)^\top, \end{aligned}$$

and this equals

$$\begin{pmatrix} 1 & -5 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ -1 \end{pmatrix} = (5, -1)^\top.$$

5. Prove the following useful facts about eigenvalues:

(a) Eigenvalues of any upper or lower triangular matrix A are the diagonal entries of A

Solution. Let us denote the diagonal elements of A by $\{a_{11}, a_{22}, a_{33}, \ldots, a_{nn}\}$. Using induction on the size of the matrix, it is easy to show by directly computing the determinant through Laplace expansion that the determinant of any triangular (or diagonal) matrix is the product of its diagonal elements. Thus the characteristic polynomial for A is:

$$\det(A - \lambda I) = (a_{11} - \lambda)(a_{22} - \lambda)\cdots(a_{nn} - \lambda),$$

so the eigenvalues are the a_{ii}'s.

(b) Show that if λ is an eigenvalue of A then λ^k is an eigenvalue of A^k for $k \in \mathbf{N}$

Solution. We use induction to show not only that λ^k is an eigenvalue of A^k, but also that any eigenvector v corresponding to the eigenvalue λ for A also corresponds to λ^k for A^k. The base step ($k = 1$) is trivial. For the induction step, assume $Av = \lambda v$ and $A^k v = \lambda^k v$. Now consider $A^{k+1}v$:

$$A^{k+1}v = A^k(Av) = A^k(\lambda v) = \lambda(A^k v) = \lambda(\lambda^k v) = \lambda^{k+1}v$$

(c) Show that if λ is an eigenvalue of the invertible matrix A then $1/\lambda$ is an eigenvalue of A^{-1}.

Solution. $Av = \lambda v$. Premultiply both sides by A^{-1}:

$$A^{-1}Av = A^{-1}\lambda v \Rightarrow v = \lambda A^{-1}v \Rightarrow A^{-1}v = (1/\lambda)v$$

6. Let A be the $n \times n$ matrix which has zeros on the main diagonal and ones everywhere else. Find the eigenvalues and eigenspaces of A and compute $\det(A)$.

   Solution. Let denote i the $n-$dimensional column vector of 1s. Note that matrix A has a very simple representation in terms of i $A = i \centerdot i^\top - I$, where I is $n \times n$ identity matrix and $\cdot \centerdot \cdot$ denotes an inner product of two vectors. To find eigenvalues and eigenvectors of A set $Ax = \lambda x$ for some non-zero vector x. We have

   $$i \centerdot i^\top x - x = (i^\top x)i - x = \lambda x$$

   which yields $(i^\top x)i = (\lambda - 1)x$. Because $(i^\top x)$ is just a number, eigenvector x must be either perpendicular or parallel to i. In the latter case, without any loss of generality, we can assume that $x = i$, thus, $i \centerdot i^\top i = (\lambda - 1)i$, which implies that $\lambda = n - 1$. This gives a $1-$ dimensional eigenspace, spanned by i with eigenvalue $\lambda = n - 1$.

   In the former case, since x is perpendicular to i, we have $i \centerdot x = 0$ and x must be in the $(n-1)-$dimensional null space of rank one matrix of ones, $i \centerdot i^\top$. thus

   $$Ax = (i \centerdot i^\top - I)x = -Ix = -x.$$

   Thus, it is easy to see that the eigenvalue associated with this eigenspace is -1 and its multiplicity is $n - 1$. Finally, determinant is just the product of eigenvalues, so we have $\det(A) = (-1)^{n-1}(n - 1)$.

7. The Supremum Norm on $L(\mathbf{R}^k, \mathbf{R}^k)$

   (a) Compute the norms of the following matrices:

   $$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \ B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \ AB = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$$

   Solution. There are multiple ways to find the norm of a matrix. In this example, the most straightforward approach is to use the fact that $||A||$ is equal to the square root of the largest eigenvalue of $A^T A$. In fact, things get even better: since each of these matrices is symmetric, we know that $A^T A = A^2$, and also from problem right above we know that for any eigenvalue λ of A, λ^2 is an eigenvalue of $A^T A$. So all we really need to do is find the eigenvalues of A is pick the one with the largest absolute value. Doing this either by inspection or by finding the characteristic polynomial yields that the absolute value of the largest eigenvalue of A is 3. So $||A|| = 3$. Using the same approach, the largest eigenvalue of B is 2. So $||B|| = 2$.
   Finally notice that $C = 3B \Longrightarrow ||C|| = 3||B|| \Longrightarrow ||C|| = 6$.

(b) Prove that for any $A, B \in L(\mathbf{R}^k, \mathbf{R}^k)$ we have the following inequality:

$$\|AB\| \leq \|A\| \, \|B\|$$

Show that *non-strict inequality* is really what we need by finding a pair of matrices such that the inequality is strict, and another pair of nonzero matrices such that equality holds.

Solution. First note that, if $||B|| = 0$, then B is the zero matrix and the weak inequality holds trivially (with equality). Otherwise, the proof follows below. The "trick" to get from line 4 to line 5 takes advantage of the fact that B maps elements of $\mathbf{R}^k$ back into $\mathbf{R}^k$, so any vector in the image of B must also be in $\mathbf{R}^k$; thus, we have

$$\sup_{Bx \neq 0} \frac{||A(Bx)||}{||Bx||} \leq \sup_{x \neq 0} \frac{||Ax||}{||x||}$$

Now,

$$\begin{aligned}
||AB|| &= \sup_{\{x \neq 0\}} \frac{||AB(x)||}{||x||} \\
&= \sup_{\{x:Bx \neq 0\}} \frac{||ABx||}{||x||} \cdot \frac{||Bx||}{||Bx||} \\
&= \sup_{\{x:Bx \neq 0\}} \frac{||ABx||}{||Bx||} \cdot \frac{||Bx||}{||x||} \\
&\leq \sup_{\{x:Bx \neq 0\}} \frac{||A(Bx)||}{||Bx||} \cdot \sup_{\{x \neq 0\}} \frac{||Bx||}{||x||} \\
&\leq \sup_{\{x \neq 0\}} \frac{||Ax||}{||x||} \cdot \sup_{\{x \neq 0\}} \frac{||Bx||}{||x||} \\
&\leq ||A|| \, ||B||.
\end{aligned}$$

For the second part of the question, the following matrices generate the strict inequality:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}.$$

Omitting the work, we can find that $||A|| = \sqrt{\frac{3+\sqrt{5}}{2}}$ and $||B|| = 4$. Thus,

$$AB = \begin{bmatrix} 4 & 4 \\ 1 & 3 \end{bmatrix} \text{ and } ||AB|| = \sqrt{\frac{7+\sqrt{17}}{2}},$$

and we have $||AB|| < ||A|| \cdot ||B||$.

For an example of equality, simply let A be the identity matrix, and B be any non-zero matrix of the same dimension as A. $||A|| = 1$ and $AB = B$, so $||AB|| = ||B||$ and the equality holds.

(c) Prove that the subset of $L(\mathbf{R}^k, \mathbf{R}^k)$ consisting of all invertible linear operators is *open* under the topology induced by the supremum norm. You may want to employ the following steps:

i. Step 1: Let $A, B \in L(\mathbf{R}^k, \mathbf{R}^k)$. Show that if A is invertible and B satisfies:

$$\|B - A\| \, \|A^{-1}\| < 1$$

Then B is one-to-one.

Solution. All of the hard work here is in step 1. First, choose some invertible operator $A \in L(\mathbf{R}^k, \mathbf{R}^k)$ and observe that:

$$||A^{-1}|| = \sup_{\{y \neq 0\}} \frac{||A^{-1}y||}{||y||} = \sup_{\{Ax \neq 0\}} \frac{||A^{-1}Ax||}{||Ax||} \geq \frac{||A^{-1}Ax||}{||Ax||} \forall x \in \mathbf{R}^k$$

Note that the y can be interchanged with the Ax without decreasing the supremum since A is an invertible matrix, so given any y, $\exists x$ such that $Ax = y$. Now, because for any vector x, $||x|| = ||A^{-1}Ax||$, it follows that: $||x|| \leq ||A^{-1}|| \cdot ||Ax||$. Using this fact, for any operator B satisfying the inequality in the hypothesis, we may write

$$\begin{aligned} ||x|| &\leq ||A^{-1}|| \, ||Ax|| \\ &\leq ||A^{-1}|| \, ||(A - B + B)x|| \\ &\leq ||A^{-1}|| \, (||(A - B)x|| + ||Bx||) \end{aligned}$$

For the last inequality we have merely used the triangle inequality for the Euclidean norm. Next, we apply the fact that $||(A - B)x|| \leq ||A - B|| \cdot ||x||$. Using the above string of inequalities, this then implies

$$\begin{aligned} ||x|| - ||A^{-1}|| \cdot ||(A - B)x|| &\leq ||A^{-1}|| \cdot ||Bx|| \\ ||x|| - ||A^{-1}|| \cdot ||A - B|| \cdot ||x|| &\leq ||A^{-1}|| \cdot ||Bx|| \\ &\Rightarrow ||x|| \cdot \frac{1 - ||A^{-1}|| \cdot ||A - B||}{||A^{-1}||} \leq ||Bx|| \end{aligned}$$

By hypothesis, $1 > ||A^{-1}|| \cdot ||A - B||$. Hence, the left-hand side of the above inequality is strictly greater than 0, whenever $x \neq \mathbf{0}$. This implies that $||Bx|| > 0$, if $x \neq \mathbf{0}$. Hence, $\ker(B) = \{\mathbf{0}\}$ and B is a one-to-one linear map, completing step 1.

ii. Step 2: Show that if B is one-to-one, then B is onto (and hence invertible).

Solution. Next, since $B : \mathbf{R}^k \to \mathbf{R}^k$, we know by the rank-nullity theorem that $\dim(\ker(B)) + \text{rank}(B) = k$. Since $\ker(B) = \{\mathbf{0}\}, \Rightarrow \text{rank}(B) = k$. Thus, B is also onto, and B is an invertible linear map. This completes step 2.

iii. Step 3: Show that there is a ball with center at A comprised entirely of invertible operators.

Solution. We may now restate our hypothesis as follows: Given some invertible $A \in L(\mathbf{R}^k, \mathbf{R}^k)$, we have that $\forall B \in L(\mathbf{R}^k, \mathbf{R}^k)$ such that $||B - A|| < \frac{1}{||A^{-1}||}$, B is also invertible. It follows that the ball of radius $1/||A^{-1}||$ around an invertible linear map A consists entirely of invertible linear maps. Hence, for all invertible operators $A \in L(\mathbf{R}^k, \mathbf{R}^k)$, we can find an $\varepsilon > 0$ (specifically, $\varepsilon = 1/||A^{-1}||$), such that the ε-ball around A contains only invertible operators. Therefore, the space of all invertible linear maps from $\mathbf{R}^k$ to $\mathbf{R}^k$ is an open subspace of the space $L(\mathbf{R}^k, \mathbf{R}^k)$.