Econ 204 – Problem Set 4

Due Tuesday, August 12

1. Similarly as it's defined in class, let $C([0,1])$ be the set of all continuous functions whose domain is the unit interval $[0,1]$ and range is $\mathbb{R}$. Let Φ be the subset consisting of all real polynomials (whose domain is restricted to the unit interval) of degree at most two:

$$\Phi \equiv \{ a + bx + cx^2 \mid a, b, c \in \mathbb{R} \}$$

Note that the set $C([0,1])$ is a vector space over the field of real numbers and the subset Φ is a proper subspace.

(a) Are the vectors $\{ x, (x^2-1), (x^2+2x+1) \}$ linearly independent over $\mathbb{R}$?

Solution Apply the usual test for independence of vectors. Solve for A, B, and C such that

$$Ax + B(x^2 - 1) + C(x^2 + 2x + 1) = 0$$

Equating like powers of x we obtain the following system in the three unknowns:

$$B + C = 0$$

$$A + 2C = 0$$

$$C - B = 0$$

$\Rightarrow C = B = -B \Rightarrow C = B = 0 \Rightarrow A = 0.$
Thus, the three vectors are linearly independent over $\mathbb{R}$.

(b) Find a Hamel basis for the subspace Φ.

Solution Clearly, $\{1, x, x^2\}$ is linearly independent and spans Φ, so it is a Hamel basis and $\dim \Phi = 3$. Also, since the set of vectors in (a) is linearly independent and contains three elements, it is a basis.

What is the dimension of Φ ? Show that $C([0,1])$ is not finite dimensional!

Solution The dimensions are three and ∞, respectively. To see that the dimension of Θ is infinite note that the set of vectors $\{1, x, x^2, x^3, ...\}$ form a linearly independent set over $\mathbb{R}$. Since Θ contains an infinite linearly indpendent set of vectors and the number of linearly independent elements of a vector space cannot exceed the dimension, the dimension of Θ must be infinite.

2. Let be λ a given eigenvalue of A. Let be the eigenspace corresponding to λ the set of the eigenvectors corresponding to λ. Prove that the eigenspace of A for a given eigenvalue is a vectorspace.

   Solution Given A and an eigenvalue for A, λ, the eigenspace of A is the set of all vectors v such that $Av = \lambda v$. Call this set V, and consider $x, y \in V$, and $\alpha \in \mathbb{R}$. We need to show that $x + y \in V$, and that $\alpha x \in V$.

   First, consider $A(x+y) = Ax + Ay = \lambda x + \lambda y = \lambda(x+y)$. So $A(x+y) = \lambda(x+y)$, and $x + y \in V$.

   Next, consider $A(\alpha x) = \alpha Ax = \alpha\lambda x = \lambda\alpha x$. So $A(\alpha x) = \lambda(\alpha x)$, and $\alpha x \in V$.

3. Let T be an invertible linear transformation. Prove that its inverse is a linear transformation.

   Solution Let be $T : X \to Y$ and $y_1, y_2 \in Y$, S the inverse of T. Need to show that $S(\alpha y_1 + \beta y_2) = \alpha S(y_1) + \beta S(y_2)$. Note that since T is invertible there exists some $x_1, x_2 \in X$ such that $T(x_1) = y_1$ and $T(x_2) = y_2$, hence $S(y_1) = x_1$ and $S(y_2) = x_2$. Then using that T is linear

$$
\begin{aligned}
S(\alpha y_1 + \beta y_2) &= S(\alpha T(x_1) + \beta T(x_2)) = \\
&= S(T(\alpha x_1) + T(\beta x_2)) = \\
&= S(T(\alpha x_1 + \beta x_2)) = \\
&= \alpha x_1 + \beta x_2 = \\
&= \alpha S(y_1) + \beta S(y_2)
\end{aligned}
$$

4. Let V have finite dimension greater than 1. Prove whether or not the set of non-invertible operators is a subspace of $L(V, V)$.

   Solution Nope, not a subspace. Fix $\dim(V) = n$, let $v = (v_1, ..., v_n)$ be in V, and define T and S by $Tv = (v_1, ..., v_{n-1}, 0)$, $Sv = (0, ..., 0, v_n)$. Then both T and S are non-invertible but $T + S$ has $(T + S)v = Tv + Sv = (v_1, ..., v_{n-1}, 0) + (0, ..., 0, v_n) = (v_1, ..., v_n) = v$. Thus $T + S$ is the identity mapping, which is invertible, and hence the set in question is not closed under addition.

5. Let A be an nxn matrix with n equal eigenvalues. Show that A is diagonalizable iff A is already diagonal.

   Solution Since $A \cdot \lambda I_{nxn} = \lambda I_{nxn} \cdot A$. If A is diagonalizable, there exists an S such that: $A = S^{-1} \cdot \lambda I_{nxn} \cdot S$, where λ is the one eigenvalue (with multiplicity n) of A. But then λI_{nxn} is scalar, hence $A = S^{-1} \cdot \lambda I_{nxn} \cdot S = S^{-1} S \cdot \lambda I_{nxn} = \lambda I_{nxn}$.

6. Suppose that V is finite dimensional and $T, S \in L(V, V)$. Prove that TS is invertible if and only if both T and S are invertible.

   Solution Assume that TS is invertible. We will first check that S is invertible. Note that by the Rank-Nullity Theorem it suffices to check that $\text{Ker}(S) = 0$. If $\exists w \in V, w \neq 0$, then we find: $TS(w) = T(0) = 0$. Hence, TS has non-zero kernel, a contradiction. Thus, S is invertible. If T is not invertible put $v \in V$ with $Tv = 0$ $(v \neq 0)$. Since S is invertible, it is surjective. Thus, we can find a $w \in V$ such that $S(w) = v$, which implies that $TS(w) = T(v) = 0$. This, again, contradicts the invertibility of TS. We now prove the converse. Assuming that T, S are both invertible we wish to check that TS is invertible. We again check the sufficient condition that $\text{Ker}(TS) = 0$. To see that this is the case, we note that if $w \in \text{Ker}(TS) \Rightarrow S(w) = 0$ or, putting $v = S(w) \neq 0$, $T(v) = 0$. Thus, if both $\text{Ker}(T)$ and $\text{Ker}(S) = 0$, then $\text{Ker}(TS) = 0$.

7. Prove that λ is an eigenvalue of a matrix A iff it is an eigenvalue of the transpose of A.

   Solution This is the result of the facts that first A and A^T has the same rank. And second that $(A - \lambda I)^T = A^T - \lambda A^T$ then $det(A - \lambda I) = 0$ iff $det(A^T - \lambda I^T) = 0$.