

Economics 204, Fall 2012  
Problem Set 5  
Due Friday, August 10 in Lecture

1. (a) Let  $f$  be defined for all real  $x$ , and suppose that

$$|f(x) - f(y)| \leq (x - y)^2$$

for all  $x, y \in \mathbb{R}$ . Prove that  $f$  is constant.

- (b) Let the real-valued function  $f$  on the open subset  $U$  of  $\mathbb{R}$  be differentiable at the point  $x_0 \in U$ . If  $\alpha, \beta \in \mathbb{R}$ , compute

$$\lim_{h \rightarrow 0} \frac{f(x_0 + \alpha h) - f(x_0 + \beta h)}{h}.$$

2. (a) If

$$a_0 + \frac{a_1}{2} + \cdots + \frac{a_{n-1}}{n} + \frac{a_n}{n+1} = 0,$$

where  $a_0, \dots, a_n$  are real constants, prove that the equation

$$a_0 + a_1x + \cdots + a_{n-1}x^{n-1} + a_nx^n = 0$$

has at least one real root between 0 and 1.

- (b) Assume that  $f : [0, \infty) \rightarrow \mathbb{R}$  is differentiable for all  $x > 0$ , and  $f'(x) \rightarrow 0$  as  $x \rightarrow \infty$ . Prove

$$\lim_{x \rightarrow \infty} [f(x+1) - f(x)] \rightarrow 0.$$

3. (a) Find the fourth-order Taylor expansion of

$$f(x) = \frac{1-x}{1+x}$$

around  $-2$ .

- (b) Find the second-order Taylor expansion of

$$g(x, y) = 7xy - y^2 - 4x^2 + x - 2y + 1$$

around  $(x, y) = (-1/2, 1/3)$ .

- (c) Find the second-order Taylor expansion of

$$h(x, y) = y \ln(xy) + e^{xy}$$

around  $(1, 1)$ .

4.  $z$  can be implicitly defined as a function of  $x$  and  $y$  by the equation  $z^3 - 2xz + y = 0$  with  $z(1, 1) = 1$ . Find the second-degree Taylor expansion of  $z$  around  $(x_0, y_0) = (1, 1)$ .
5. Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be a  $C^1$  function, with  $f(2, -1) = -1$ . Set  $G(x, y, u) = f(x, y) + u^2$  and  $H(x, y, u) = ux + 3y^3 + u^3$ . The equations

$$\begin{aligned} G(x, y, u) &= 0, \\ H(x, y, u) &= 0 \end{aligned}$$

have a solution  $(x_0, y_0, u_0) = (2, -1, 1)$ .

- (a) What conditions on  $Df(x, y)$  ensure that there are  $C^1$  functions  $x = g(y)$  and  $u = h(y)$  (defined on an open set in  $\mathbb{R}$  that contains  $y_0 = -1$ ) which satisfy both equations, such that  $g(-1) = 2$  and  $h(-1) = 1$ ?
- (b) Under the conditions of part (a), and assuming that  $Df(2, -1) = (1, -3)$ , find  $g'(-1)$  and  $h'(-1)$ .
6. Use the Implicit Function Theorem to prove the Inverse Function Theorem (i.e. take the assumptions of the Inverse Function Theorem as given, relate them to the assumptions of the Implicit Function Theorem and use the conclusions of the Implicit Function Theorem to derive the conclusions of the Inverse Function Theorem).