

Economics 204  
Fall 2013  
Problem Set 5  
Due Friday, August 16 in Lecture

1. Let  $A$  and  $B$  be  $n \times n$  matrices such that  $A^2 = A$  and  $B^2 = B$ . Suppose that  $A$  and  $B$  have the same rank. Prove that  $A$  and  $B$  are similar.
2. Identify which of the following matrices can be diagonalized and provide the diagonalization. If you claim that a diagonalization does not exist, prove it.

$$\begin{bmatrix} 2 & -3 \\ 2 & -5 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 3 \\ 1 & 1 & -1 \end{bmatrix}, \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{bmatrix}$$

3. Let  $A$  and  $B$  be  $n \times n$  matrices. Prove or disprove each of the following statements:
  - (a) If  $A$  and  $B$  are diagonalizable, so is  $A + B$ .
  - (b) If  $A$  and  $B$  are diagonalizable, so is  $A \cdot B$ .
  - (c) If  $A^2 = A$  then  $A$  is diagonalizable.
4. Prove that if  $f : \mathbf{R} \rightarrow \mathbf{R}$  is differentiable, then  $\text{Im}(f')$  is an interval (possibly a singleton).
5. Some practice with implicit function theorem
  - (a) Define  $f : \mathbf{R}^3 \rightarrow \mathbf{R}$  by

$$f(x, y_1, y_2) = x^2 y_1 + e^x + y_2.$$

Show that  $f(0, 1, -1) = 0$  and  $D_1 f(0, 1, -1) \neq 0$ . Thus, there exists a differentiable function  $g$  in some neighborhood of  $(1, -1)$  of  $\mathbf{R}^2$ , such that  $g(1, -1) = 0$  and

$$f(g(y_1, y_2), y_1, y_2) = 0.$$

Compute  $Dg(1, -1)$ .

- (b) Let  $x = x(y, z)$ ,  $y = y(x, z)$  and  $z = z(x, y)$  be functions that are implicitly defined by equation  $F(x, y, z) = 0$ . Prove that

$$\frac{\partial x}{\partial y} \frac{\partial y}{\partial z} \frac{\partial z}{\partial x} = -1.$$

6. Let  $f : \mathbf{R} \rightarrow \mathbf{R}$  be  $\mathcal{C}^1$  function and let

$$\begin{aligned}u &= f(x) \\v &= -y + xf(x).\end{aligned}$$

If  $f'(x_0) \neq 0$ , show that this transformation is locally invertible in the neighborhood of  $(x_0, y_0)$  and that the inverse has the form

$$\begin{aligned}x &= g(u) \\y &= -v + ug(u).\end{aligned}$$

7. Using Taylor's formula approximate following two functions

$$f(x, y) = (1 + x)^m(1 + y)^n \quad \text{and} \quad f(x, y) = \frac{\cos x}{\sin y}.$$

up to the second order terms. Estimate approximation error.