# Econ 204 – Problem Set 5

Due Friday, August 15, 2014

1. Let $A_1$, $A_2$, ... and $L$ be square matrices from $\mathbb{R}^{n\times n}$. We say that $\{A_k\}$ converges to $L$ if

$$\lim_{k\to\infty}(A_k)_{ij} = L_{ij}\ \forall 1\le i\le n, 1\le j\le n$$

i.e. each element $(A_k)_{ij}$ converges to $L_{ij}$. Prove or provide a counterexample to the following:

   a) If $\lim_{m\to\infty} A^m$ exists, then every eigenvalue of $A$ satisfies $-1<\lambda\le 1$.
   b) If $A$ is diagonalizable and every eigenvalue of $A$ satisfies $-1<\lambda\le 1$, then $\lim_{m\to\infty} A^m$ exists.
   c) If every eigenvalue of $A$ satisfies $-1<\lambda\le 1$, then $\lim_{m\to\infty} A^m$ exists.

2. In section we saw that the set of invertible matrices, $\Omega(\mathbb{R}^{n\times n}) = \{A\in\mathbb{R}^{n\times n} : A^{-1} \text{ exists}\}$, is an open subset of $\mathbb{R}^{n\times n}$. Prove that $\Omega(\mathbb{R}^{n\times n})$ is dense in $\mathbb{R}^{n\times n}$.

3. For the following functions, determine at what points the derivative exists, and if the derivative function is continuous (you may use that the derivative of $\sin x$ is $\cos x$):

$$f(x)=\begin{cases} x\cdot\sin\left(\frac{1}{x}\right) & \text{if } x\ne 0\\ 0 & \text{if } x=0\end{cases},\quad g(x)=\begin{cases} x^2\cdot\sin\left(\frac{1}{x}\right) & \text{if } x\ne 0\\ 0 & \text{if } x=0\end{cases}$$

4. Let $f:\mathbb{R}\to\mathbb{R}$ be a differentiable function. Prove that $f'(\mathbb{R})$, the image of the derivative function, is an interval (possibly a singleton).

5. If $a_0+\frac{1}{2}a_1+\cdots+\frac{1}{n}a_{n-1}+\frac{1}{n+1}a_n=0$, where $a_0,\ldots,a_n$ are real constants, prove that the equation

$$a_0+a_1x+\cdots+a_{n-1}x^{n-1}+a_nx^n=0$$

has at least one real root between 0 and 1.

6. Assume that $f:[0,\infty)\to\mathbb{R}$ is differentiable for all $x>0$, and $f'(x)\to 0$ as $x\to\infty$. Prove

$$\lim_{x\to\infty}[f(x+1)-f(x)]\to 0.$$