Economics 204
Fall 2012
Problem Set 5 Suggested Solutions

1. (a) Let f be defined for all real x, and suppose that

$$|f(x) - f(y)| \leq (x - y)^2$$

for all $x, y \in \mathbb{R}$. Prove that f is constant.

(b) Let the real-valued function f on the open subset U of $\mathbb{R}$ be differentiable at the point $x_0 \in U$. If $\alpha, \beta \in \mathbb{R}$, compute

$$\lim_{h \to 0} \frac{f(x_0 + \alpha h) - f(x_0 + \beta h)}{h}.$$

Solution:

(a) We know $|f(x) - f(y)| \leq (x - y)^2 = |x - y|^2$. Therefore for all $x \neq y$ we have

$$\left| \frac{f(x) - f(y)}{x - y} \right| = \frac{|f(x) - f(y)|}{|x - y|} \leq |x - y|.$$

Fix some $y \in \mathbb{R}$. Then, letting $x \to y$ we get

$$0 \leq \lim_{x \to y} \left| \frac{f(x) - f(y)}{x - y} \right| \leq \lim_{x \to y} |x - y| = 0.$$

So $\lim_{x \to y} \left| \frac{f(x)-f(y)}{x-y} \right| = 0$ and therefore $\lim_{x \to y} \frac{f(x)-f(y)}{x-y} = 0$. This implies $f'(y) = 0$ for all $y \in \mathbb{R}$.

But this means that f is a constant function. To see that, note that the existence of the derivative of f at all points of its domain means that we can apply the mean value theorem to get $f(b) - f(a) = f'(c)(b - a)$ for all $a, b \in \mathbb{R}$ and some c between them. But $f'(c) = 0$ and so $f(a) = f(b)$.

(b)

$$\begin{aligned}
&\lim_{h\to 0}\frac{f(x_0+\alpha h)-f(x_0+\beta h)}{h}\\
=&\lim_{h\to 0}\frac{f(x_0+\alpha h)-f(x_0)+f(x_0)-f(x_0+\beta h)}{h}\\
=&\lim_{h\to 0}\left(\frac{f(x_0+\alpha h)-f(x_0)}{h}-\frac{f(x_0+\beta h)-f(x_0)}{h}\right)\\
=&\lim_{h\to 0}\left(\alpha\frac{f(x_0+\alpha h)-f(x_0)}{\alpha h}-\beta\frac{f(x_0+\beta h)-f(x_0)}{\beta h}\right)\\
=&\alpha\lim_{h\to 0}\left(\frac{f(x_0+\alpha h)-f(x_0)}{\alpha h}\right)-\beta\lim_{h\to 0}\left(\frac{f(x_0+\beta h)-f(x_0)}{\beta h}\right)\\
=&\alpha f'(x_0)-\beta f'(x_0)\\
=&(\alpha-\beta)f'(x_0),
\end{aligned}$$

where the penultimate equality follows from the fact that $h \to 0$ implies $\alpha h \to 0$ and $\beta h \to 0$.

2. (a) If

$$a_0+\frac{a_1}{2}+\cdots+\frac{a_{n-1}}{n}+\frac{a_n}{n+1}=0,$$

where $a_0,\ldots,a_n$ are real constants, prove that the equation

$$a_0+a_1x+\cdots+a_{n-1}x^{n-1}+a_nx^n=0$$

has at least one real root between 0 and 1.

(b) Assume that $f : [0,\infty) \to \mathbb{R}$ is differentiable for all $x > 0$, and $f'(x) \to 0$ as $x \to \infty$. Prove

$$\lim_{x\to\infty}[f(x+1)-f(x)]\to 0.$$

Solution:

(a) Let $f(x) = a_0x + \frac{a_1}{2}x^2 + \cdots + \frac{a_n}{n+1}x^{n+1}$. This function is clearly differentiable everywhere and so, applying the mean value theorem we have $f(1)-f(0) = f'(c)(1-0)$ for some $c \in (0,1)$. Clearly $f(0) = 0$ and by the given equation we also have $f(1) = 0$. Thus we must have $f'(c) = 0$. Note that

$$f'(c)=a_0+a_1c+\cdots+a_{n-1}c^{n-1}+a_nc^n,$$

which implies that c is a root of the equation.

(b) Fix $g(x) = f(x+1) - f(x)$. Since f is differentiable for all $x > 0$ we can apply the mean value theorem. It states that for all $x > 0$ there exists some $c_x \in (x, x+1)$ such that $g(x) = f'(c_x)(x+1-x) = f'(c_x)$. But we have $\lim_{c\to\infty} f'(c) = 0$. So, since $c_x > x$, we have

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} f'(c_x) = \lim_{c\to\infty} f'(c) = 0,$$

which is what we wanted to show.

3. (a) Find the fourth-order Taylor expansion of

$$f(x) = \frac{1-x}{1+x}$$

around -2.

(b) Find the second-order Taylor expansion of

$$g(x,y) = 7xy - y^2 - 4x^2 + x - 2y + 1$$

around $(x,y) = (-1/2, 1/3)$.

(c) Find the second-order Taylor expansion of

$$h(x,y) = y\ln(xy) + e^{xy}$$

around $(1,1)$.

Solution:

(a)

$$\begin{aligned}
f(x) &= \frac{1-x}{1+x} \Rightarrow f(-2) = -3 \\
f'(x) &= \frac{-(1+x)-(1-x)}{(1+x)^2} = -\frac{2}{(1+x)^2} \Rightarrow f'(-2) = -2 \\
f''(x) &= \frac{4}{(1+x)^3} \Rightarrow f''(-2) = -4 \\
f'''(x) &= -\frac{12}{(1+x)^4} \Rightarrow f'''(-2) = -12 \\
f^{(4)}(x) &= \frac{48}{(1+x)^5} \Rightarrow f^{(4)}(-2) = -48.
\end{aligned}$$

Then the fourth-order expansion is

$$
\begin{aligned}
f(x) &= f(-2) + f'(-2)(x+2) + \frac{1}{2}f''(-2)(x+2)^2 \\
&+ \frac{1}{6}f'''(-2)(x+2)^3 + \frac{1}{24}f^{(4)}(-2)(x+2)^4 \\
&= -3 - 2(x+2) - 2(x+2)^2 - 2(x+2)^3 - 2(x+2)^4 \\
&= -3 - 2(x + 2 + x^2 + 4x + 4 + x^3 + 6x^2 \\
&+ 12x + 8 + x^4 + 8x^3 + 24x^2 + 32x + 16) \\
&= -3 - 2(x^4 + 9x^3 + 31x^2 + 49x + 30) \\
&= 63 - 98x - 62x^2 - 18x^3 - 2x^4.
\end{aligned}
$$

(b) The function g is a second-degree polynomial. Therefore its second-order Taylor expansion at any point is g itself.

(c)

$$
\begin{aligned}
h(x,y) &= y\ln(xy) + e^{xy} \Rightarrow h(1,1) = e \\
Dh(x,y) &= \begin{pmatrix} y^2\frac{1}{xy} + ye^{xy} & \ln(xy) + xy\frac{1}{xy} + xe^{xy} \end{pmatrix} \\
&= \begin{pmatrix} y/x + ye^{xy} & \ln(xy) + 1 + xe^{xy} \end{pmatrix} \\
\Rightarrow Dh(1,1) &= \begin{pmatrix} 1+e & 1+e \end{pmatrix} \\
D^2h(x,y) &= \begin{pmatrix} -y/x^2 + y^2e^{xy} & 1/x + e^{xy} + xye^{xy} \\ 1/x + e^{xy} + xye^{xy} & 1/y + x^2e^{xy} \end{pmatrix} \\
\Rightarrow D^2h(1,1) &= \begin{pmatrix} -1+e & 1+2e \\ 1+2e & 1+e \end{pmatrix}
\end{aligned}
$$

Then the second-order expansion is

$$
\begin{aligned}
h(x,y) &= h(1,1) + Dh(1,1)\begin{pmatrix} x-1 \\ y-1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} x-1 & y-1 \end{pmatrix} D^2h(1,1)\begin{pmatrix} x-1 \\ y-1 \end{pmatrix} \\
&= e + \begin{pmatrix} 1+e & 1+e \end{pmatrix}\begin{pmatrix} x-1 \\ y-1 \end{pmatrix} \\
&+ \frac{1}{2}\begin{pmatrix} x-1 & y-1 \end{pmatrix}\begin{pmatrix} -1+e & 1+2e \\ 1+2e & 1+e \end{pmatrix}\begin{pmatrix} x-1 \\ y-1 \end{pmatrix} \\
&= e + (1+e)(x-1) + (1+e)(y-1) \\
&+ \frac{1}{2}[(e-1)(x-1)^2 + 2(1+2e)(x-1)(y-1) + (1+e)(y-1)^2] \\
&= \frac{e-1}{2}x^2 + (2e+1)xy + \frac{e+1}{2}y^2 + (1-2e)x - (2e+1)y + (2e-1)
\end{aligned}
$$

4. z can be implicitly defined as a function of x and y by the equation $z^3 - 2xz + y = 0$ with $z(1,1) = 1$. Find the second-degree Taylor expansion of z around $(x_0, y_0) = (1,1)$.

   Solution: Let $(x_0, y_0, z_0) = (1,1,1)$. Since the function $F(x,y,z) = z^3 - 2xz + y$ is C^1, $F(x_0, y_0, z_0) = 0$, and $\det(D_z F(x_0, y_0, z_0)) = 3z_0^2 - 2x_0 = 1 \neq 0$, we can apply the Implicit Function Theorem. In particular, we can compute the derivative of $z(x,y)$ at (x_0, y_0) to be

   $$\begin{aligned} Dz(x_0, y_0) &= -[D_z F(x_0, y_0, z_0)]^{-1}[D_{(x,y)} F(x_0, y_0, z_0)] \\ &= -\frac{1}{3z_0^2 - 2x_0} \begin{pmatrix} -2z_0 & 1 \end{pmatrix}. \end{aligned}$$

   Plugging in $(x_0, y_0, z_0) = (1,1,1)$, we get $Dz(x_0, y_0) = \begin{pmatrix} 2 & -1 \end{pmatrix}$. Notice that this implies $\frac{\partial z(x_0,y_0)}{\partial x} = 2$ and $\frac{\partial z(x_0,y_0)}{\partial y} = -1$. Let's denote these two partial derivatives by z_x and z_y respectively. We can then compute the Hessian of $z(x,y)$ at (x_0, y_0) to be

   $$\begin{aligned} D^2 z(x_0, y_0) &= \begin{pmatrix} \frac{2z_x(3z_0^2 - 2x_0) - 2z_0(6z_0 z_x - 2)}{(3z_0^2 - 2x_0)^2} & \frac{2z_y(3z_0^2 - 2x_0) - 2z_0(6z_0 z_y)}{(3z_0^2 - 2x_0)^2} \\ \frac{6z_0 z_x - 2}{(3z_0^2 - 2x_0)^2} & \frac{6z_0 z_y}{(3z_0^2 - 2x_0)^2} \end{pmatrix} \\ &= \begin{pmatrix} -16 & 10 \\ 10 & -6 \end{pmatrix}. \end{aligned}$$

   Then the second-order Taylor expansion of $z(x,y)$ around $(x_0, y_0) = (1,1)$ becomes

   $$\begin{aligned} & z(x_0, y_0) + Dz(x_0, y_0) \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}^\top D^2 z(x_0, y_0) \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} \\ =& 1 + \begin{pmatrix} 2 & -1 \end{pmatrix} \begin{pmatrix} x - 1 \\ y - 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} x - 1 \\ y - 1 \end{pmatrix}^\top \begin{pmatrix} -16 & 10 \\ 10 & -6 \end{pmatrix} \begin{pmatrix} x - 1 \\ y - 1 \end{pmatrix} \\ =& 1 + 2(x-1) - (y-1) - 8(x-1)^2 + 10(x-1)(y-1) - 3(y-1)^2. \end{aligned}$$

5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a C^1 function, with $f(2,-1) = -1$. Set $G(x,y,u) = f(x,y) + u^2$ and $H(x,y,u) = ux + 3y^3 + u^3$. The equations

   $$\begin{aligned} G(x,y,u) &= 0, \\ H(x,y,u) &= 0 \end{aligned}$$

   have a solution $(x_0, y_0, u_0) = (2, -1, 1)$.

(a) What conditions on $Df(x, y)$ ensure that there are C^1 functions $x = g(y)$ and $u = h(y)$ (defined on an open set in $\mathbb{R}$ that contains $y_0 = -1$) which satisfy both equations, such that $g(-1) = 2$ and $h(-1) = 1$?

(b) Under the conditions of part (a), and assuming that $Df(2, -1) = (1, -3)$, find $g'(-1)$ and $h'(-1)$.

Solution:

(a) Consider the function $J : \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$J(x, y, u) = (G(x, y, u), H(x, y, u)).$$

Because G and H are C^1 functions, it follows that J is a C^1 function that satisfies $J(2, -1, 1) = (0, 0)$. Therefore, the Implicit Function Theorem implies that as long as

$$\det D_{(x,u)}J(2, -1, 1) = \det \begin{pmatrix} D_x f(2, -1) & 2 \\ 1 & 5 \end{pmatrix} = 5D_x f(2, -1) - 2 \neq 0,$$

there is an open neighborhood $A \subset \mathbb{R}$ of the point $y_0 = -1$ and C^1 functions $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ (this follows because a function is C^1 if and only if its component functions are C^1) such that $g(-1) = 2$, $h(-1) = 1$ and $J(g(y), y, h(y)) = (0, 0)$ for all $y \in A$. Of course, this is equivalent to both $G(g(y), y, h(y)) = 0$ and $H(g(y), y, h(y)) = 0$ for all $y \in A$.

(b) Assuming that $Df(2, -1) = (1, -3)$, the Implicit Function Theorem implies that

$$\begin{aligned}
\begin{pmatrix} g'(-1) \\ h'(-1) \end{pmatrix} &= -[D_{(x,u)}J(2, -1, 1)]^{-1}[D_y J(2, -1, 1)] \\
&= \frac{1}{3} \begin{pmatrix} -5 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} -3 \\ 9 \end{pmatrix} \\
&= \begin{pmatrix} 11 \\ -4 \end{pmatrix},
\end{aligned}$$

so that $g'(-1) = 11$ and $h'(-1) = -4$.

6. Use the Implicit Function Theorem to prove the Inverse Function Theorem (i.e. take the assumptions of the Inverse Function Theorem as

given, relate them to the assumptions of the Implicit Function Theorem and use the conclusions of the Implicit Function Theorem to derive the conclusions of the Inverse Function Theorem).

Solution: Suppose the assumptions of the Inverse Function Theorem hold: i.e. suppose $X \subseteq \mathbb{R}^n$ is open, $f : X \to \mathbb{R}^n$ is C^1 on X, $x_0 \in X$ and $\det Df(x_0) \neq 0$.

Define a function $F : X \times \mathbb{R}^n \to \mathbb{R}^n$ by $F(x, y) = f(x) - y$ so that $F(x_0, f(x_0)) = f(x_0) - f(x_0) = 0$. Notice that $D_x F(x, y) = Df(x)$ so that $\det D_x F(x_0, f(x_0)) \neq 0$. Finally, both X and $\mathbb{R}^n$ are open subsets of $\mathbb{R}^n$. So we can apply the Implicit Function Theorem to the function F to conclude that there exists some open neighborhoods U' of x_0 and V of $f(x_0)$ such that:

- For all $y \in V$ there is unique $x \in U'$ such that $F(x, y) = 0$ or, in other words, $f(x) - y = 0 \Leftrightarrow f(x) = y$. So we can construct a function $g : V \to U'$, which is C^1 by the Implicit Function Theorem.

  Let $U = U' \cap f^{PI}(V)$ (where we denote the preimage of f by f^{PI} to avoid confusion with f's inverse). Since f is continuous and V is open, U is also open as the intersection of open sets. It is clear that $g(V) \subset U$ since for all $y \in V$ we have $g(y) \in U'$ and, since $g(y) = x$ so that $f(x) = y$, $g(y) \in f^{PI}(V)$. Conversely, $U \subset g(V)$ since $x \in U$ implies $x \in f^{PI}(V)$ and hence there is some $y \in V$ such that $f(x) = y$ and so $g(y) = x$. So $g(V) = U$ and it follows that $g : V \to U$ is onto.

  Additionally, $g : V \to U$ is also one-to-one since if $g(y) = g(y')$ then $y = f(g(y)) = f(g(y')) = y'$, which implies $y = y'$. Finally, note that g is clearly f's (local) inverse: if $x \in U$ and $f(x) = y$ then, since g is one-to-one and onto, $y \in V$ and so $g(y) = x$. Since g is one-to-one and onto and is the inverse of f, then $f : U \to V$ is one-to-one and onto as well.

- By the Implicit Function Theorem, we also have

  $$\begin{aligned} Df^{-1}(f(x_0)) = Dg(f(x_0)) &= -[D_x F(x_0, f(x_0))]^{-1}[D_y F(x_0, f(x_0))] \\ &= -[Df(x_0)]^{-1}(-1) \\ &= [Df(x_0)]^{-1}, \end{aligned}$$

  where the second line follows from our construction of F.

- Finally, if f is C^k, it is clear by F's construction that so is F. Hence by the Implicit Function Theorem f's inverse g is also C^k.