Problem 1.

Let $A_1, A_2, \ldots$ and L be square matrices from $\mathbb{R}^{n\times n}$. We say that $\{A_k\}$ converges to L if

$$\lim_{k\to\infty}(A_k)_{ij} = L_{ij}\ \forall 1\le i\le n, 1\le j\le n$$

i.e. each element $(A_k)_{ij}$ converges to L_{ij}. Prove or provide a counterexample to the following:

a) If $\lim_{m\to\infty} A^m$ exists, then every eigenvalue of A satisfies $|\lambda|<1$ or $\lambda=1$.

b) If A is diagonalizable and every eigenvalue of A satisfies $|\lambda|<1$ or $\lambda=1$, then $\lim_{m\to\infty} A^m$ exists.

c) If every eigenvalue of A satisfies $|\lambda|<1$ or $\lambda=1$, then $\lim_{m\to\infty} A^m$ exists.

Solution

First we prove the following:

Theorem. *If* $\lim_{m\to\infty} A^m = L$ *exists, then* $\lim_{m\to\infty} PA^m = PL$ *and* $\lim_{m\to\infty} A^m Q = LQ$.

Proof. Take the ij^{th} element:

$$\begin{aligned}
\lim_{m\to\infty}(PA^m)_{ij} &= \lim_{m\to\infty}\left(\sum_{k=1}^n P_{ik}(A_m)_{kj}\right)\\
&= \sum_{k=1}^n P_{ik}\lim_{m\to\infty}(A_m)_{kj}\\
&= \sum_{k=1}^n P_{ik}L_{kj} = (PL)_{ij}
\end{aligned}$$

The other direction is the same. □

a) True. Let $L=\lim_{m\to\infty} A^m$. If λ is an eigenvalue and v a corresponding eigenvector of A then inductively we have $A^m v = A^{m-1}Av = A^{m-1}(\lambda v) = \ldots = \lambda^m v$. Hence λ^m is an eigenvalue and v a corresponding eigenvector of A^m. Then from the above theorem, $Lv = \lim_{m\to\infty}(A^m v) = \lim_{m\to\infty}(\lambda^m v) = (\lim_{m\to\infty}\lambda^m)\, v$, which exists iff $|\lambda|<1$ or $\lambda=1$.

b) True. If A is diagonalizable then $A = Q^{-1}\Lambda Q$ where Λ is a diagonal matrix of eigenvalues, and

$$A^m = (Q^{-1}\Lambda Q)\cdots(Q^{-1}\Lambda Q) = Q^{-1}\Lambda^m Q$$

Since $|\lambda|<1$ or $\lambda=1$, $\lim_{m\to\infty}\lambda_i^m$ exists for $i=1\ldots,n$. Hence $\lim_{m\to\infty}\Lambda^m$ exists, and from the above theorem (applied twice) we have $\lim_{m\to\infty} A^m$ exists.

c) False. Consider $B = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$. Then $B^m = \begin{pmatrix}1 & m\\ 0 & 1\end{pmatrix}$.

Correction: Originally I wrote "every eigenvalue satisfies $-1<\lambda\le 1$" however real-valued matrices can still have complex eigenvalues; as originally written, (a) is false.

Problem 2.

In section we saw that the set of invertible matrices, $\Omega(\mathbb{R}^{n\times n}) = \{A \in \mathbb{R}^{n\times n} : A^{-1} \text{ exists}\}$, is an open subset of $\mathbb{R}^{n\times n}$. Prove that $\Omega(\mathbb{R}^{n\times n})$ is dense in $\mathbb{R}^{n\times n}$.

Solution

What we want to show is that every singular matrix A is a limit point of the set of all invertible matrices. So fix some singular matrix A and consider the the characteristic polynomial of A: $g(t) = \det(A - tI)$. From what we saw in section, g is a continuous function, and we know that g is zero only at eigenvalues: $g(\lambda_i) = 0$ for $i = 1\dots, n$ (not necessarily distinct). Since $\det(A) = 0$ we have that 0 is an eigenvalue, so choose $\lambda = \operatorname{argmin}(|\lambda_i| : \lambda_i \neq 0)$. Then take the sequence $t_n = \frac{1}{n}$. We can find some N such that $n > N$ implies $t_n < |\lambda|$, so $g(t_n) \neq 0$. But this says the matrix $A - t_n I$ is invertible. Further, using the metric d induced by the norm defined on the space of all $n \times n$ matrices:

$$\begin{aligned} d(A, A - t_n I) &= \|A - (A - t_n I)\| \\ &= \|t_n I\| \\ &= |t_n| \to 0 \end{aligned}$$

Hence $A - t_n I$ is a sequence of invertible matrices that approaches A. So A is a limit point of the set of all invertible matrices.

Remark: Note that the above holds for any norm we can define on the space of all $n \times n$ matrices.

Problem 3.

For the following functions, determine at what points the derivative exists, and if the derivative function is continuous (you may use that the derivative of $\sin x$ is $\cos x$):

$$f(x) = \begin{cases} x \cdot \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}, \quad g(x) = \begin{cases} x^2 \cdot \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Solution

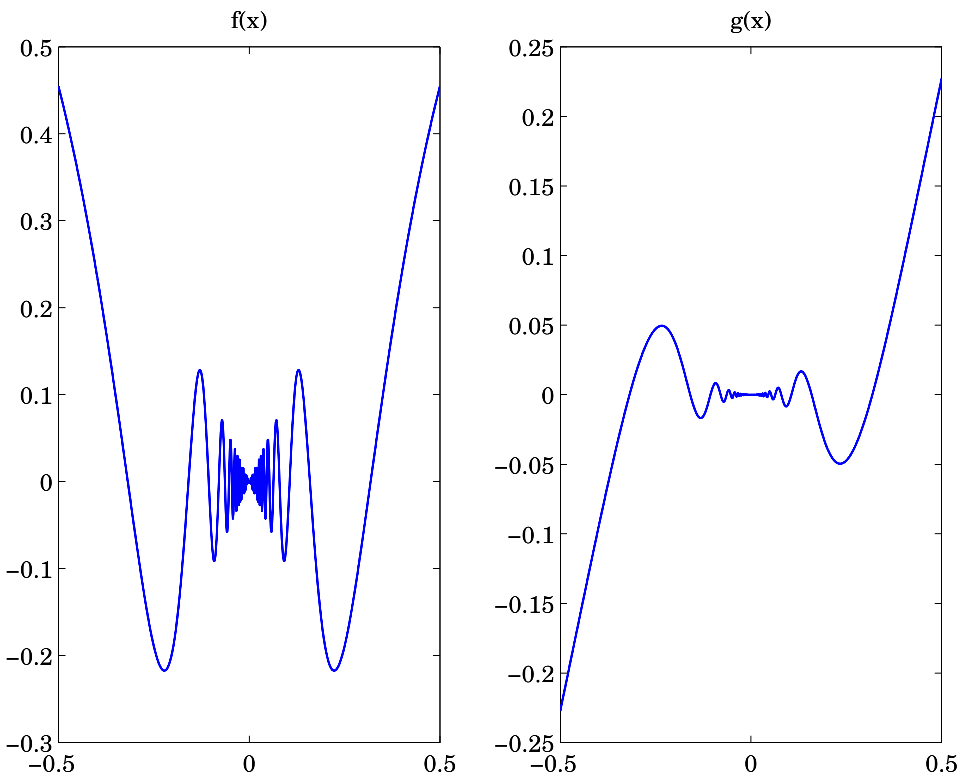
For $x \neq 0$ we can find the derivatives of f and g using the simple properties of derivatives:

$$f'(x) = \sin\frac{1}{x} - \frac{1}{x}\cos\frac{1}{x}, \quad g'(x) = 2x\sin\frac{1}{x} - \cos\frac{1}{x}$$

At $x = 0$ we use directly the definition of the derivative. Note that for $h \neq 0$ we have

$$\frac{f(h) - f(0)}{h} = \sin\frac{1}{h}, \quad \frac{g(h) - g(0)}{h} = h\sin\frac{1}{h}$$

Since $\lim_{h\to 0}\sin\frac{1}{h}$ is not defined, $f'(0)$ is not defined. However, note $\left|h\sin\frac{1}{h}\right| \leq |h|$ so $g'(0) = 0$ and the derivative of g exists everywhere. But $\lim_{x\to 0}\cos\frac{1}{x}$ is not defined, so $\lim_{x\to 0} g'(x) \neq g'(0)$, ie g' is not continuous at $x = 0$.



Problem 4.

Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Prove that $f'(\mathbb{R})$, the image of the derivative function, is an interval (possibly a singleton).

Solution

To prove the claim it suffices to show that for any $a, b \in f'(\mathbb{R})$ with $a < b$, and any $c \in (a, b)$, we have $c \in f'(\mathbb{R})$. Note that if there are no two distinct values, since f is differentiable we have $f'(\mathbb{R}) \neq \varnothing$; so $f'(\mathbb{R}) = \{c\}$ and we're done (this occurs if f is a constant function).

Choose x_1, x_2 such that $f'(x_1) = a, f'(x_2) = b$, and assume without loss of generality that $x_1 < x_2$. Define the function $g : \mathbb{R} \to \mathbb{R}$ where $g(x) = f(x) - cx$. g is also a differentiable function with $g'(x) = f'(x) - x$. This implies that g is continuous, hence by the Extreme Value Theorem g attains its minimum (and maximum) on the closed interval $[x_1, x_2]$.

Now note that $g'(x_1) = a - c < 0$, which says

$$\lim_{h \to 0} \frac{g(x_1 + h) - g(x_1)}{h} < 0$$

So for some $h' > 0$ we have that for every $0 < \varepsilon < h'$, $\frac{g(x_1+\varepsilon)-g(x_1)}{\varepsilon} < 0 \implies g(x_1+\varepsilon)-g(x_1) < 0 \implies g(x_1 + \varepsilon) < g(x_1)$, so $g(x_1)$ is not a minimum of $g([x_1, x_2])$. A similar argument shows that since $g'(x_2) = b - c > 0$, $g(x_2)$ is not a minimum either. So g attains its minimum at some $x_0 \in (x_1, x_2)$, and the same argument implies that $g'(x_0) = 0$. Thus we have $f'(x_0) = c \iff c \in f'(\mathbb{R})$.

Remark: Note that we can't use the Intermediate Value Theorem since we can't assume f' is a continuous function.

Problem 5.

If $a_0 + \frac{1}{2}a_1 + \cdots + \frac{1}{n}a_{n-1} + \frac{1}{n+1}a_n = 0$, where $a_0, \ldots, a_n$ are real constants, prove that the equation

$$a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} + a_n x^n = 0$$

has at least one real root between 0 and 1.

Solution

Let $f(x) = a_0 x + \frac{a_1}{2}x^2 + \cdots + \frac{a_n}{n+1}x^{n+1}$. This function is clearly differentiable everywhere, with $f'(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} + a_n x^n$. Applying the mean value theorem we have $f(1) - f(0) = f'(c)(1-0)$ for some $c \in (0,1)$. Clearly $f(0) = 0$ and from how the coefficients were constructed we also have $f(1) = 0$. Thus we must have $f'(c) = 0$.

Problem 6.

Assume that $f : [0, \infty) \to \mathbb{R}$ is differentiable for all $x > 0$, and $f'(x) \to 0$ as $x \to \infty$. Prove

$$\lim_{x\to\infty} [f(x + 1) - f(x)] \to 0.$$

Solution

Define $g(x) = f(x+1) - f(x)$. Since f is differentiable we can apply the mean value theorem: for all $x > 0$ we can find some $c_x \in (x, x + 1)$ such that $g(x) = f'(c_x)(x + 1 - x) = f'(c_x)$. But we have $f'(x) \to 0$, and since $c_x > x$, we have

$$\lim_{x\to\infty} [f(x + 1) - f(x)] = \lim_{x\to\infty} g(x) = \lim_{x\to\infty} f'(c_x) = \lim_{c_x\to\infty} f'(c_x) = 0$$