Economics 204
Fall 2011
Problem Set 6
Due Monday, August 13 in Exam

1. Call a function $f : X \to \mathbf{R}$ defined on a convex subset X of Euclidean space *quasi-convex* if for all $x,\, y \in X$ and all $\lambda \in [0,1]$ we have $f(\lambda x + (1-\lambda)y) \leq \max\{f(x),\, f(y)\}$. Similarly, call a function f quasi-concave if $-f$ is a quasi-convex function.

   Lets assume that f and g be continuous functions on $\mathbf{R}^2$ such that for all fixed $x \in \mathbf{R}^2$ $f(\cdot, x_2)$ is quasi-concave and $g(x_1,\, \cdot)$ is quasi-convex. In addition, assume that $f([0,\, 1], x_2) \cap \mathbf{R}_+ \neq \emptyset$ and $g(x_1, [0,\, 1]) \cap \mathbf{R}_- \neq \emptyset$. Prove that there is $x \in [0,\, 1]^2$ such that
$$f(x) \geq 0 \geq g(x).$$
   (Hint: Use Kakutani's Fixed Point Theorem)

2. Let $C \subseteq \mathbf{R}^n$ be a closed, convex subset with the additional property that $C \cap \mathbf{R}^n_+ = \{0\}$. Show that $C + \omega \cap \mathbf{R}^n_+$ is compact for any $\omega \in \mathbf{R}^n$. (Hint: Use the Separating Hyperplane Theorem.)

3. Show that $\mathbf{R}^2$ cannot be the countable union of the range of $\mathcal{C}^1$ functions from $\mathbf{R}$ to $\mathbf{R}^2$. (Hint: Use Sard's Theorem.)

4. Consider the following inhomogeneous second order linear differential equation
$$x''(t) - 2x'(t) + x(t) = \sin(t)$$
   (a) Write down the corresponding homogeneous equation.
   (b) Find the general solution of the homogeneous equation.
   (c) Find a particular solution of the original inhomogeneous equation satisfying the initial condition $x(0) = (1)$ and $x'(0) = 0$.
   (d) Find the general solution of the original inhomogeneous equation.