Economics 204
Fall 2013
Problem Set 6
Due Wednesday August 21 at 9am in the final exam room

1. Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be C^1 function. Prove that the image of Lebesgue measure zero set must have measure zero as well. (Hint: You can use without proof that countable union of Lebesgue measure zero sets has Lebesgue measure zero).

2. Lets call a vector $\pi \in \mathbf{R}^n$ a probability distribution for the states of the world if and only if $\sum_{i=1}^n \pi_i = 1$ and $\pi_i \geq 0$ for all $i = 1, 2, \ldots, n$, i.e. π_i is the probability of state i occurring. Suppose that there are n states of the world and two traders (trader 1 and trader 2). Corresponding to each trader is a closed, convex, and compact set of *prior* probability distributions denoted by Π_1 and Π_2. A *trade* is a vector $f \in \mathbf{R}^n$ corresponding to the net transfer trader 1 receives in each state of the world (so that $-f$ corresponds to the net transfer that trader 2 receives in each state of the world). A trade $f \in \mathbf{R}^n$ is *agreeable* if
$$\inf_{\pi \in \Pi_1} \sum_{i=1}^n \pi_i f_i > 0 \qquad \text{and} \qquad \inf_{\pi \in \Pi_2} \sum_{i=1}^n \pi_i(-f_i) > 0.$$
Prove that there exists an agreeable trade if and only if there is no common prior ($\Pi_1 \cap \Pi_2 = \emptyset$).

3. Consider the set $X = [0, 1]^2$ and the correspondence $\Psi(x) : X \to 2^X$, defined by
$$\Psi(x) = \text{argmax}_{y \in X} || \, y - x||$$
   (a) Draw a picture, showing the images under Ψ of $x_0 = (0, 0)$, $x_1 = (\frac{1}{2}, 0)$ and $x_2 = (\frac{1}{2}, \frac{1}{2})$.
   (b) At what points $x \in X$ is Ψ convex-valued? Compact-valued? Upper hemi-continuous? (No proofs needed, but give precise definitions of these concepts and explain your answers. Also, you may use without proof Berge's Maximum Theorem which states that for two metric spaces (X, d) and (Θ, σ), $f : X \times \Theta \to \mathbf{R}$ a continuous function, $\Psi : \Theta \to 2^X$ compact-valued and continuous correspondence, and $\psi^*(\theta)$ and $g^*(\theta)$ defined as follows
$$\psi^*(\theta) = \text{argmax}\{f(x, \theta) : x \in \Psi(\theta)\} \text{ for all } \theta \in \Theta$$
$$g^*(\theta) = \max\{f(x, \theta) : x \in \Psi(\theta)\} \text{ for all } \theta \in \Theta$$
   we have $\psi^* : \Theta \to 2^X$ is a compact-valued, upper hemi-continuous and closed at θ and $g^*(\theta) : \Theta \to \mathbf{R}$ is continuous at θ).
   (c) Which (if any) of the Kakutani's Theorem requirements are met by Ψ?

(d) Find the (possibly empty) set of fixed points under Ψ.

(e) Consider the correspondence $\Phi : X \to 2^X$ where, for each $x \in X$, $\Phi(x)$ is the convex hull of $\Psi(x)$. Redo parts $(a) - (d)$ for $\Phi(x)$.

4. Consider the second order linear differential equation given by $y'' = -y - y'$.

   (a) Show how this equation can be rewritten as the following *first* order linear differential equation of two variables y_1 and y_2.

   (b) Describe the solutions of the first order system (verbally) by analyzing the matrix A.

   (c) In a phase diagram, show the behavior of the system using the previous analysis and by solving for $y_1'(t) = 0$ and $y_2'(t) = 0$.

   (d) Give the solution of the system when $y_1(t_0) = 0$ and $y_2'(t_0) = 1$.

5. Consider the following inhomogeneous second order linear differential equation

$$x''(t) + 3x'(t) - 4x(t) = 2e^{-4t}$$

   (a) Write down the corresponding homogeneous equation.

   (b) Find the general solution of the homogeneous equation.

   (c) Find a particular solution of the original inhomogeneous equation satisfying the initial condition $x(0) = (1)$ and $x'(0) = 0$.

   (d) Find the general solution of the original inhomogeneous equation.