Econ 204 – Problem Set 6

Due Tuesday, August 20; before exam

1. Calculate the second and third order Taylor expansion of $(1 + 2x - 3y)^2$ around the point $(0, 0)$. Calculate the difference between the value of the function and the expansions.

2. Consider the following equations:

$$u = \frac{x}{x^2 + y^2}, \quad v = \frac{y}{x^2 + y^2}, \quad x^2 + y^2 > 0.$$

   (a) For $(u, v) = (1/2, 1/2)$, find a pair of values (x_0, y_0) that satisfy the equations.

   (b) Describe either verbally or graphically what this transformation does. Bonus given for colorful metaphors.

   (c) Show that the above transformation implicitly defines a function in the neighborhood of (x_0, y_0) (in the sense that for every pair of values (u, v) near $(1/2, 1/2)$, there is just one corresponding pair of (x, y) values.

   (d) Compute the Jacobian of the implicit function.

3. Prove that there exist functions $u, v : \mathbb{R}^4 \longrightarrow \mathbb{R}$, continuously differentiable on some open neighborhood around the point $(x, y, z, w) = (2, 1, -1, 2)$ such that $u(2, 1, -1, 2) = 4$ and $v(2, 1, -1, 2) = 3$ and the equations

$$u^2 + v^2 + w^2 = 29 \text{ and } \frac{u^2}{x^2} + \frac{v^2}{y^2} + \frac{w^2}{z^2} = 17$$

   both hold for all (x, y, z, w) in that neighborhood.

4. Let $E = \{(x, y) : 0 < y < x\}$ and set $f(x, y) = (x + y, xy)$ for $(x, y) \in E$.

   (a) Prove f is one-to-one from E onto $\{(s, t) : s > 2\sqrt{t},\ t > 0\}$ and find a formula for $f^{-1}(s, t)$.

   (b) Use the inverse function theorem to compute $D(f^{-1})(f(x, y))$ for $x \neq y$.

   (c) Compare the two expressions for $D(f^{-1})(f(x, y))$ that you derived directly of using the Implicit Function Theorem

5. Consider the following system of first order differential equations:

$$\begin{aligned} \dot{x} &= x^{1/4} - y \\ \dot{y} &= y[\tfrac{3}{2}x^{-2/3} - \tfrac{1}{10}] \end{aligned}$$

(a) Plot the $\dot{x} = 0$ and $\dot{y} = 0$ loci for $x > 0$ in a phase diagram. Show the steady state, the direction of motion, and the approximate location of the stable and unstable arms.

(b) Linearize the system using a Taylor-series expansion around the $x > 0$ steady state. Write down the linearized equations.

(c) Plot a phase diagram for the linearized system and compare the behavior at the steady state of the two systems.

(d) Give the general solution of the linearized system.

6. Consider the second order linear differential equation given by $y'' = -y - y'$.

(a) Show how this equation can be rewritten as the following *first* order linear differential equation of two variables:

$$\bar{x}'(t) = A\bar{x}(t),$$

where $A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$ and $\bar{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$.

(b) Describe the solutions of the first order system (verbally) by analyzing the matrix A.

(c) In a phase diagram, show the behavior of the system using the previous analysis and by solving for $x_1'(t) = 0$ and $x_2'(t) = 0$.

(d) Give the solution of the system when $x_1(t_0) = 0$ and $x_2'(t_0) = 1$.