Economics 204
Fall 2012
Problem Set 6 Suggested Solutions

1. Call a function $f : X \to \mathbf{R}$ defined on a convex subset X of Euclidean space *quasi-convex* if for all $x,\, y \in X$ and all $\lambda \in [0,1]$ we have $f(\lambda x + (1-\lambda)y) \leq \max\{f(x),\, f(y)\}$. Similarly, call a function f quasi-concave if $-f$ is a quasi-convex function.

   Lets assume that f and g be continuous functions on $\mathbf{R}^2$ such that for all fixed $x \in \mathbf{R}^2$ $f(\cdot, x_2)$ is quasi-concave and $g(x_1,\, \cdot)$ is quasi-convex. In addition, assume that $f([0,\, 1], x_2) \cap \mathbf{R}_+ \neq \emptyset$ and $g(x_1, [0,\, 1]) \cap \mathbf{R}_- \neq \emptyset$. Prove that there is $x \in [0,\, 1]^2$ such that
$$f(x) \geq 0 \geq g(x).$$
   (Hint: Use Kakutani's Fixed Point Theorem)

   Solution. Lets define following self-correspondences on a unit interval
$$\Psi_f(x_2) = \{x_1 \in [0,1] :\ f(x_1,\, x_2) \geq 0\}$$
$$\Psi_g(x_1) = \{x_2 \in [0,1] :\ g(x_1,\, x_2) \leq 0\}$$
   Clearly, these two correspondences are non-empty by our assumption. Now, lets define correspondence Γ on $[0,\, 1]^2$ as $\Gamma(x) = \Psi_f(x_2) \times \Psi_g(x_1)$. Again, it is trivially non-empty. Note that by construction $x \in [0,\, 1]^2$ is a fixed point of the correspondence Γ if and only if our desired result holds. To show its existence, we use Kakutani's Fixed Point Theorem.

   Firstly, note that our correspondence Γ has a closed graph, because f and g are continuous (and graph of continuous function is closed). In addition, Γ has a compact range, implying immediately that Γ is upper hemi-continuous. To prove the convex-valuedness of Γ we use the alternative characterization of quasi-concavity and quasi-convexity, that upper- and lower-contour sets (respectively) of those functions are convex sets.[1] Finally, Γ is clearly compact-valued by construction, and we get the result we desire.

2. Let $C \subseteq \mathbf{R}^n$ be a closed, convex subset with the additional property that $C \cap \mathbf{R}^n_+ = \{0\}$. Show that $C + \omega \cap \mathbf{R}^n_+$ is compact for any $\omega \in \mathbf{R}^n$. (Hint: Use the Separating Hyperplane Theorem.)

   Solution. $(C + \omega) \cap \mathbf{R}^n_+$ is the intersection of two closed sets, hence it is closed. Suppose it is not bounded. Then we can find a sequence $c_m \in C$ such that $c_m + \omega \in \mathbf{R}^n_+$ and $|c_m + \omega| \to \infty$, so $|c_m| \to \infty$. Since C is convex and $0 \in C$, $d_m = \frac{c_m}{|c_m|} \in C$. Since $|d_m| = 1$ for all m, we can choose a convergent subsequence $d_{m_k} \to c \in \mathbf{R}^n$. Since $|d_m| = 1$, $|c| = 1$, so $c \neq 0$. Since C is

[1] You can look up these results for instance in Mas-Collel Winston and Green, p. 933-34.

closed, $c \in C$. Since $c_m + \omega \in \mathbf{R}^n_+$, $c_m \geq -\omega$, so $d_m \geq -\frac{\omega}{|c_m|} \to 0$, so $c \geq 0$, so $c \in C \cap \mathbf{R}^n_+$. Since $c \neq 0$, $C \cap \mathbf{R}^n_+ \neq \{0\}$, contradiction. Thus, $(C + \omega) \cap \mathbf{R}^n_+$ is a closed and bounded subset of $\mathbf{R}^n$, hence it is compact.

3. Show that $\mathbf{R}^2$ cannot be the countable union of the range of $\mathcal{C}^1$ functions from $\mathbf{R}$ to $\mathbf{R}^2$. (Hint: Use Sard's Theorem.)

   Solution. The idea for the proof is that we have smooth $\mathcal{C}^1$ functions, mapping from lower-dimensional space to higher dimensional space (that is why we need an extra degree of differentiability, i.e. smoothness). As the result every point of the domain is a critical point of f.

   Formally, it is a direct implication of Sard's Theorem in in de la Fuente p. 215. Here $m < n$, and $X = \mathbf{R}$ is open, then the set of critical points of f is the whole $\mathbf{R}$. The theorem suggests that range $(f) = f(C_f)$ has Lebesgue measure 0. Finally, countable union of Lebesque measure null sets is a null set, while $\mathbf{R}^2$ is not measure zero. Thus, we get the result we desire.

4. Consider the following inhomogeneous second order linear differential equation

$$x''(t) - 2x'(t) + x(t) = \sin(t)$$

   (a) Write down the corresponding homogeneous equation.

   Solution.
$$x''(t) - 2x'(t) + x(t) = 0$$

   (b) Find the general solution of the homogeneous equation.

   Solution. We rewrite the second order linear homogenous equation as a system of two variables, i.e. lets define the new variable

$$\bar{x} = \begin{bmatrix} x \\ x' \end{bmatrix}.$$

   This gives us:

$$\bar{x}' = \begin{bmatrix} x' \\ x'' \end{bmatrix} = \begin{bmatrix} x' \\ 2x' - x \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ x' \end{bmatrix} = A\bar{x}.$$

   The characteristic polynomial $\chi(\lambda) = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$ so the only eigenvalue of A is 1 with multiplicity 2. From the lecture notes, we immediately know the general solution of homogenous equation must be of the form:

$$C_1 e^t + C_2 t e^t,$$

   for some constants C_1, $C_2 \in \mathbf{R}$.

(c) Find a particular solution of the original inhomogeneous equation satisfying the initial condition $x(0) = (1)$ and $x'(0) = 0$.

Solution. It is easy to check that $\frac{\cos t}{2}$ is a particular solution to the original equation, so the general solution is

$$C_1 e^t + C_2 t e^t + \frac{\cos t}{2}.$$

Using the initial conditions, we pin down $C_1 = -\frac{1}{2}$ and $C_2 = \frac{1}{2}$,

(d) Find the general solution of the original inhomogeneous equation.

Solution. Since we know form the class that the general solution of the original inhomogeneous equation is just the sum of general solutions to the homogenous equation and a particular solution to inhomogeneous equation, we get

$$\frac{1}{2}\left(te^t - e^t + \cos t\right).$$