Economics 204
Fall 2013
Problem Set 6 Suggested Solutions

1. Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be C^1 function. Prove that the image of Lebesgue measure zero set must have measure zero as well. (Hint: You can use without proof that countable union of Lebesgue measure zero sets has Lebesgue measure zero).

   Solution. Let $A \subset \mathbf{R}^n$ and let $\mu(\cdot)$ denote the Lebesgue measure of set, so that we have $\mu(A) = 0$. Let $\mathcal{C}$ be the set of all critical points of f and define following collection of sets $B_\varepsilon = \{x \in \mathbf{R}^n : |\det f'(x)| > 0\}$ for all arbitrarily small $\varepsilon > 0$. Observe that $A \cap B_\varepsilon \subset A$ implies $\mu(A \cap B_\varepsilon) = 0$. Moreover, because $|\det f'(x)| > 0$ on B_ε (i.e. f is diffeomorphism on B_ε) we can apply integral change of variables formula

$$\int_{f(A\cap B_\varepsilon)} dx = \int_{A\cap B_\varepsilon} |\det f'|\, dt = 0$$

   to conclude that $\mu(f(A \cap B_\varepsilon)) = 0$. Now we use the hint that countable union of Lebesgue measure zero sets has Lebesgue measure zero to get that

$$\mu\left(\cup_{n=1}^{\infty} f(A \cap B_{\frac{1}{n}})\right) = 0.$$

   Finally, $f(A \cap C) \subset f(C)$ so by Sard's Theorem it must have null measure as well, and we get the result we desire as

$$A = \left(\cup_{n=1}^{\infty} (A \cap B_{\frac{1}{n}})\right) \cup C.$$

2. Lets call a vector $\pi \in \mathbf{R}^n$ a probability distribution for the states of the world if and only if $\sum_{i=1}^n \pi_i = 1$ and $\pi_i \geq 0$ for all $i = 1, 2, \ldots, n$, i.e. π_i is the probability of state i occurring. Suppose that there are n states of the world and two traders (trader 1 and trader 2). Corresponding to each trader is a closed, convex, and compact set of *prior* probability distributions denoted by Π_1 and Π_2. A *trade* is a vector $f \in \mathbf{R}^n$ corresponding to the net transfer trader 1 receives in each state of the world (so that $-f$ corresponds to the net transfer that trader 2 receives in each state of the world). A trade $f \in \mathbf{R}^n$ is *agreeable* if

$$\inf_{\pi\in\Pi_1} \sum_{i=1}^n \pi_i f_i > 0 \qquad \text{and} \qquad \inf_{\pi\in\Pi_2} \sum_{i=1}^n \pi_i(-f_i) > 0.$$

   Prove that there exists an agreeable trade if and only if there is no common prior ($\Pi_1 \cap \Pi_2 = \emptyset$).

Solution. Suppose there exists an agreeable trade $f \in \mathbf{R}^n$ and that $\Pi_1 \cap \Pi_2 \neq \emptyset$. Choose any $\pi^* \in \Pi_1 \cap \Pi_2$, then

$$\sum_{i=1}^{n} \pi_i^* f_i \geq \inf_{\pi \in \Pi_1} \sum_{i=1}^{n} \pi_i f_i > 0.$$

But this implies that $\sum_i \pi_i^*(-f_i) = -\sum_i \pi_i^* f_i < 0$, which contradicts the fact that $\inf_{\pi \in \Pi_2} \sum_i \pi_i(-f_i) > 0$. It follows, then, that $\Pi_1 \cap \Pi_2 = \emptyset$.

Suppose that $\Pi_1 \cap \Pi_2 = \emptyset$. Because Π_1 and Π_2 are closed, convex, compact, and disjoint sets in $\mathbf{R}^n$, the Strict Separating Hyperplane Theorem guarantees the existence of a nonzero trade $f \in \mathbf{R}^n$ and some $k \in \mathbf{R}$ such that $f \cdot x > k$ for all $x \in \Pi_1$ and $f \cdot x < k$ for all $x \in \Pi_2$. This implies that the trade $g \in \mathbf{R}^n$ given by $g_i = f_i - k$ for all $i = 1, 2, \ldots, n$ satisfies

$$\inf_{\pi \in \Pi_1} \sum_{i=1}^{n} \pi_i g_i = \inf_{\pi \in \Pi_1} \sum_{i=1}^{n} (\pi_i f_i - \pi_i k) = \inf_{\pi \in \Pi_1} \sum_{i=1}^{n} \pi_i f_i - k > 0$$

and

$$\inf_{\pi \in \Pi_2} \sum_{i=1}^{n} \pi_i(-g_i) = -\sup_{\pi \in \Pi_2} \sum_{i=1}^{n} (\pi_i f_i - \pi_i k) = -\sup_{\pi \in \Pi_2} \sum_{i=1}^{n} \pi_i f_i - k > 0.$$

3. Consider the set $X = [0,1]^2$ and the correspondence $\Psi(x) : X \to 2^X$, defined by

$$\Psi(x) = \text{argmax}_{y \in X} || y - x ||$$

(a) Draw a picture, showing the images under Ψ of $x_0 = (0,0)$, $x_1 = (\frac{1}{2}, 0)$ and $x_2 = (\frac{1}{2}, \frac{1}{2})$.

Solution. Lets define following sets $E = \{(0,0), (1,0), (1,1), (0,1)\}$, $V = \{x \in X : x_1 = \frac{1}{2}, x_2 \neq \frac{1}{2}\}$, $H = \{x \in X : x_1 \neq \frac{1}{2}, x_2 = \frac{1}{2}\}$ and $C = \{(\frac{1}{2}, \frac{1}{2})\}$. Then we have

$$\begin{aligned}
\Psi(x_0) &= \Psi((0,0)) = \{(1,1)\} \\
\Psi(x_1) &= \Psi((\frac{1}{2}, 0)) = \{(1,1), (0,1)\} \\
\Psi(x_2) &= \Psi((\frac{1}{2}, \frac{1}{2})) = \{(0,0), (1,0), (1,1), (0,1)\}
\end{aligned}$$

(b) At what points $x \in X$ is Ψ convex-valued? Compact-valued? Upper hemi-continuous? (No proofs needed, but give precise definitions of these concepts and explain your answers. Also, you may use without proof Berge's Maximum Theorem which states that for two metric spaces (X, d)

and (Θ, σ), $f : X \times \Theta \to \mathbf{R}$ a continuous function, $\Psi : \Theta \to 2^X$ compact-valued and continuous correspondence, and $\psi^*(\theta)$ and $g^*(\theta)$ defined as follows

$$\psi^*(\theta) = \text{argmax}\{f(x,\theta) : x \in \Psi(\theta)\} \text{ for all } \theta \in \Theta$$
$$g^*(\theta) = \max\{f(x,\theta) : x \in \Psi(\theta)\} \text{ for all } \theta \in \Theta$$

we have $\psi^* : \Theta \to 2^X$ is a compact-valued, upper hemi-continuous and closed at θ and $g^*(\theta) : \Theta \to \mathbf{R}$ is continuous at θ).

Solution.

$$\Psi(x) \text{ is } \begin{cases} \text{singleton valued } \subset E, & x \in E \\ \text{two pont set } \subset E, & x \in V \cup H \\ \text{four pont set } = E, & x \in C \end{cases}$$

more precisely

$$\Psi(x) = \begin{cases} \{(1-x_1, 1-x_2)\} \subset E, & x \in E \\ \{(0,1),(1,1)\} \subset E, & x \in V, x_2 < \frac{1}{2} \\ \{(0,0),(1,0)\} \subset E, & x \in V, x_2 > \frac{1}{2} \\ \{(1,0),(1,1)\} \subset E, & x \in H, x_2 < \frac{1}{2} \\ \{(0,0),(0,1)\} \subset E, & x \in Y, x_2 > \frac{1}{2} \\ E & x \in C \end{cases}$$

Now, $\Psi(x)$ is convex-valued for all $x \in E$, and not convex for all $x \in X \setminus E$, so Ψ is not convex-valued.

$\Psi(x)$ is closed-valued for all $x \in X$ (as union of singletons which are closed sets), so Ψ is closed-valued.

Ψ is upper hemi-continuous as a direct implication of Berge's Theorem of Maximum.

(c) Which (if any) of the Kakutani's Theorem are met by Ψ?

Solution. Every condition in Kakutanis Theorem holds except the Ψ being convex-valued.

(d) Find the (possibly empty) set of fixed points under Ψ.

Solution. The set of fixed points for correspondence is $\emptyset$, since for all $x \in X$ $||x - x|| = 0 < ||x - y||$ for all $y \neq x$, so $x \notin \Psi(x)$ for all $x \in X$.

(e) Consider the correspondence $\Phi : X \to 2^X$ where, for each $x \in X$, $\Phi(x)$ is the convex hull of $\Psi(x)$. Redo parts $(a) - (d)$ for $\Phi(x)$.

Solution. Lets denote by the $co(A)$ a convex hull of set A. Now $\Phi = co(\Psi)$ for all $x \in X$ and clearly Φ now is convex-valued.

$$\begin{aligned}
\Phi(x_0) &= \Phi((0,0)) = co(\{(1,1)\}) = \{(1,1)\} \\
\Phi(x_1) &= \Phi((\frac{1}{2},0)) = co(\{(1,1),(0,1)\}) = [0,1] \times \{1\} \\
\Phi(x_2) &= \Phi((\frac{1}{2},\frac{1}{2})) = co(\{(0,0),(1,0),(1,1),(0,1)\}) = X.
\end{aligned}$$

Also, we have

$$\begin{aligned}
\Phi(x) &= \begin{cases} co(\{(1-x_1, 1-x_2)\}), & x \in E \\ co(\{(0,1),(1,1)\}), & x \in V, x_2 < \frac{1}{2} \\ co(\{(0,0),(1,0)\}), & x \in V, x_2 > \frac{1}{2} \\ co(\{(1,0),(1,1)\}), & x \in H, x_2 < \frac{1}{2} \\ co(\{(0,0),(0,1)\}), & x \in Y, x_2 > \frac{1}{2} \\ co(E) & x \in C \end{cases} \\
&= \begin{cases} \{(1-x_1, 1-x_2)\}, & x \in E \\ [0,1] \times \{1\}, & x \in V, x_2 < \frac{1}{2} \\ [0,1] \times \{0\}, & x \in V, x_2 > \frac{1}{2} \\ \{1\} \times [0,1], & x \in H, x_2 < \frac{1}{2} \\ \{0\} \times [0,1], & x \in Y, x_2 > \frac{1}{2} \\ [0,1] \times [0,1] & x \in C \end{cases}
\end{aligned}$$

We see that all conditions in Kakutani's Theorem are met, so $\Phi(x)$ has at least one fixed point, i.e. there is $x \in X$ such that $x^* \in \Phi(x^*)$. The set of fixed point of the correspondence $\Phi(x)$ is $\{\frac{1}{2}, \frac{1}{2}\}$.

4. Consider the second order linear differential equation given by $y'' = -y - y'$.

   (a) Show how this equation can be rewritten as the following *first* order linear differential equation of two variables y_1 and y_2.

   Solution. Write it as

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

   (b) Describe the solutions of the first order system (verbally) by analyzing the matrix A.

   Solution. The eigenvalues are $\lambda_{1,2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. Because the eigenvalues are complex, the solutions to the system spiral around the origin. Because the real parts of the solutions are both negative, the system spirals inward, converging to a steady state at the origin.

(c) In a phase diagram, show the behavior of the system using the previous analysis and by solving for $y_1'(t) = 0$ and $y_2'(t) = 0$.

Solution. See Figure 1. The $y_1' = 0$ locus is the line $y_2 = 0$, that is the horizontal axis. All path of the system cross this line vertically. Above this line, the solution moves from left to right and below it solutions move from right to left. The $y_2' = 0$ locus is the line $y_2 = -y_1$ and all paths cross this line horizontally. Above this line solutions move from up to down and below the line they move from down to up. The intersection of these two line yields a stable steady state at the origin. In summary, the solutions spiral inwards around the origin in a clockwise direction.

(d) Give the solution of the system when $y_1(t_0) = 0$ and $y_2'(t_0) = 1$.

Solution. From lectures, we know immediately the solution is of the form

$$y(t) = e^{-(t-t_0)/2}(C_1 \cos(\sqrt{3}(t-t_0)/2) + C_2 \sin(\sqrt{3}(t-t_0)/2).$$

We obtain this by simply substituting in the eigenvalues obtained above. Substituting in $t = t_0$, you get $y_0 = y(t_0) = C_1$, but we are given the initial condition that $y_0 = 0$. This means that $C_1 = 0$ and

$$y(t) = C_2 e^{-(t-t_0)/2} \sin(\sqrt{3}(t-t_0)/2).$$

Differentiating yields

$$y'(t) = -(C_2/2)e^{-(t-t_0)/2} \sin(\sqrt{3}(t-t_0)/2) + (C_2\sqrt{3}/2)e^{-(t-t_0)/2} \cos(\sqrt{3}(t-t_0)/2$$

and substituting in the second initial condition gives us

$$1 = y'(0) = C_2\sqrt{3}/2 \quad \Longrightarrow \quad C_2 = 2/\sqrt{3},$$

so

$$y(t) = (2/\sqrt{3})e^{-(t-t_0)/2} \sin(\sqrt{3}(t-t_0)/2).$$

5. Consider the following inhomogeneous second order linear differential equation

$$x''(t) + 3x'(t) - 4x(t) = 12t^2 + 2t - 1$$

(a) Write down the corresponding homogeneous equation.

Solution.

$$x''(t) + 3x'(t) - 4x(t) = 0.$$

(b) Find the general solution of the homogeneous equation.

Solution. As in the previous problem we rewrite the second order linear homogenous equation as a system of two variables, i.e. lets define the new variable

$$\bar{x} = \begin{pmatrix} x \\ x' \end{pmatrix}.$$

This gives us:

$$\bar{x}' = \begin{pmatrix} x' \\ x'' \end{pmatrix} = \begin{pmatrix} x' \\ 2x' - x \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix} = A\bar{x}.$$

The characteristic polynomial $\chi(\lambda) = \lambda^2 + 3\lambda - 4$ so we have two real eigenvalues of A, 4 and -1. From the lecture notes, we immediately know the general solution of homogenous equation must be of the form:

$$C_1 e^{4t} + C_2 e^{-t}$$

for some constants C_1, $C_2 \in \mathbf{R}$.

(c) Find a particular solution of the original inhomogeneous equation

Solution. To find a particular solution we employ the method of undetermined coefficients. It is applicable here because right hand side of our differential equation and its successive derivatives together contain only a finite number of distinct types of expression (apart from multiplicative constants).

Now, we know that the particular solution, by definition, is a value of $x(t)$ satisfying the given equation (regardless of the value of t). Note that LHS contains function $x(t)$ and its derivatives $x'(t)$ and $x''(t)$, whereas RHS contains multiples of t^2, t, and a constant. So, we suppose initially that the particular solutions would be

$$x(t) = at^2 + bt + c.$$

Differentiating we get $x'(t) = 2at + b$ and $x''(t) = 2a$. Plugging those in

$$x''(t) + 3x'(t) - 4x(t) = -4at^2 + (6a - 4b)t + 2a + 3b - 4c$$

which must be equal to $12t^2 + 2t - 1$. Matching the coefficients we get $a = -3$, $b = -5$, and $c = -5$. Thus, the desired particular solution is

$$x_p = -3t^2 - 5t - 5.$$

(d) Find the general solution of the original inhomogeneous equation satisfying the initial condition $x(0) = 1$ and $x'(0) = 0$.

Solution. Since we know form lectures that the general solution of the original inhomogeneous equation is just the sum of general solutions to the homogenous equation and a particular solution to inhomogeneous equation, we get

$$x(t) = C_1 e^{4t} + C_2 e^{-t} - 3t^2 - 5t - 5.$$

Now we use the boundary conditions to pin down constants of integration C_1 and C_2. We have $x(0) = 1$ implying $C_1 + C_2 = 1$ and $x'(0) = 0$ implying $4C_1 - C_2 = 0$. Thus, we get $C_1 = \frac{1}{5}$ and $C_2 = \frac{4}{5}$, yielding

$$x(t) = \frac{1}{5} e^{4t} + \frac{4}{5} e^{-t} - 3t^2 - 5t - 5.$$

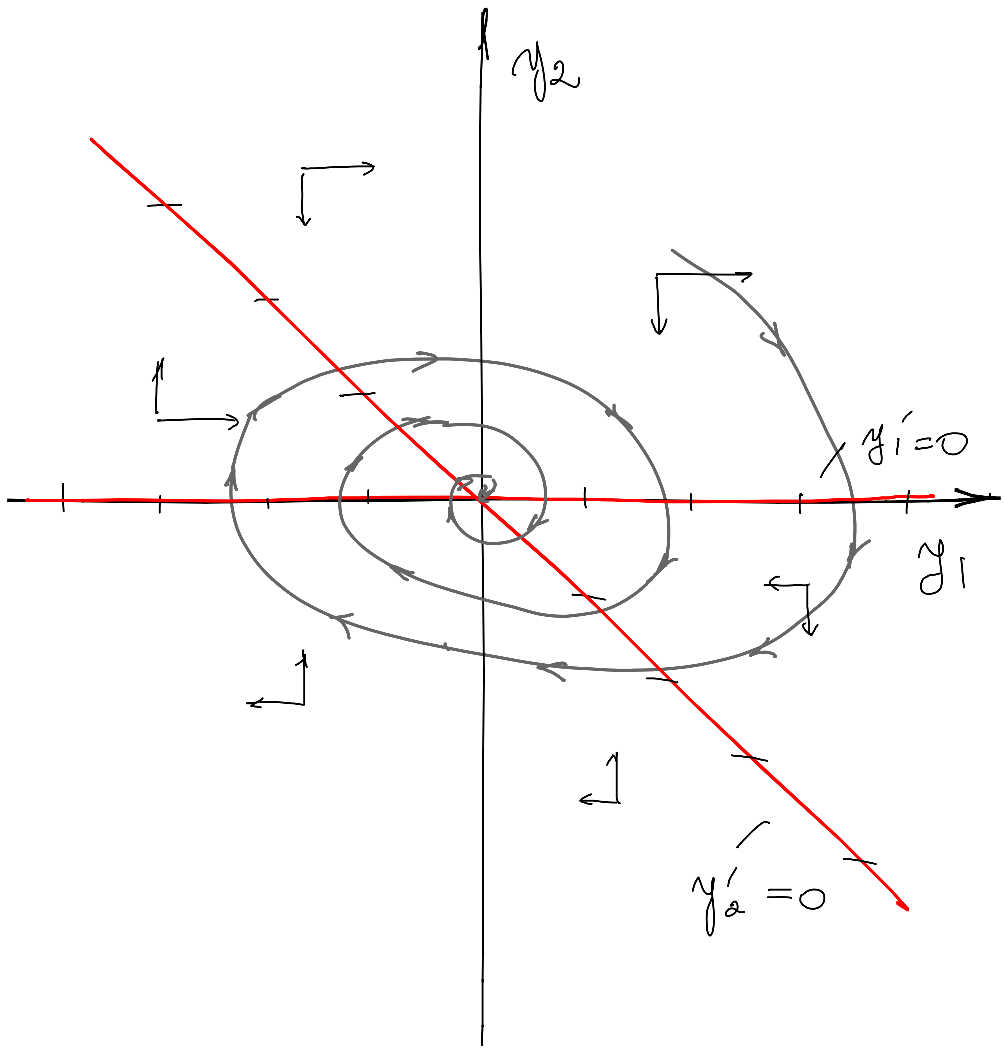

Figure 1.