Econ 204 – Problem Set 6

Due Tuesday, August 20; before exam

1. Calculate the second and third order Taylor expansion of $(1+2x-3y)^2$ around the point $(0,0)$. Calculate the difference between the value of the function and the expansions.

   Solution We want to expand $f(x,y) = (1+2x-3y)^2$ at the point $(0,0)$. Generally, a second order expansion of $f(x,y)$ at $(0,0)$ is the following:

   $$f(x,y) = f(0,0) + Df(0,0)\begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{2!}(x,y)D^2 f(0,0)\begin{pmatrix} x \\ y \end{pmatrix}$$

   Now we have to calculate some derivatives. We have:

   $$\begin{array}{ll} f_x(x,y) = 4(1+2x-3y) & f_x(0,0) = 4 \\ f_y(x,y) = -6(1+2x-3y) & f_y(0,0) = -6 \\ f_{xx}(x,y) = 8 & f_{xx}(0,0) = 8 \\ f_{xy}(x,y) = -12 & f_{xy}(0,0) = -12 \\ f_{yy}(x,y) = 18 & f_{yy}(0,0) = 18 \end{array}$$

   We also have $f(0,0) = 1$, and that all third and higher order derivatives are zero. Thus, our second order Taylor expansion should exactly equal $f(x,y)$. Plugging our derivatives in to our Taylor expansion yields:

   $$f(x,y) = 1 + (4,-6)\begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{2}(x,y)\begin{bmatrix} 8 & -12 \\ -12 & 18 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

   To verify that this Taylor expansion exactly equals $f(x,y)$, we will need to do some algebra. Carrying out the matrix multiplications above, we have:

   $$f(x,y) = 1 + 4x - 6y + 4x^2 - 12xy + 9y^2$$

   To complete the check, we take the square of $(1+2x-3y)$ to find that:

   $$f(x,y) = (1+2x-3y)^2 = 1 + 4x - 6y + 4x^2 - 12xy + 9y^2$$

   Which is exactly the same as the second order Taylor expansion.

   Since the second order expansion exactly equal the function then it equals the third order expansions as well.

2. Consider the following equations:

   $$u = \frac{x}{x^2+y^2}, \quad v = \frac{y}{x^2+y^2}, \quad x^2+y^2 > 0.$$

   (a) For $(u,v) = (1/2, 1/2)$, find a pair of values (x_0, y_0) that satisfy the equations.

   Solution The point $(x_0, y_0) = (1,1)$ satisfies the equations.

(b) Describe either verbally or graphically what this transformation does. Bonus given for colorful metaphors.

Solution Points near the origin are mapped to points very far away from the origin. Points very far away are mapped to points very close to the origin. This transformation takes $\mathbf{R}^2$, rips it open at the navel $(0,0)$ and turns it inside out. Zero becomes infinity and infinity becomes zero. Each point on a given ray starting at the origin gets mapped to another point on the same ray. Points on the unit circle remain the same.

(c) Show that the above transformation implicitly defines a function in the neighborhood of (x_0, y_0) (in the sense that for every pair of values (u,v) near $(1/2, 1/2)$, there is just one corresponding pair of (x,y) values.

Solution Define $F : \mathbf{R}^4 \to \mathbf{R}^2$ by

$$F((x,y),(u,v)) = \left(u - \frac{x}{x^2+y^2},\ v - \frac{y}{x^2+y^2} \right)$$

From our answer to part (a), we see that when $\mathbf{a} = (1,1)$ and $\mathbf{b} = (1/2,1/2)$, $F(\mathbf{a},\mathbf{b}) = (0,0)$. We compute the Jacobian (restricted to x and y) and evaluate at $(\mathbf{a},\mathbf{b})$ to check whether we can apply the Implicit Function Theorem. If the Jacobian is invertible, then we can. We find

$$DF_{xy}(\mathbf{a},\mathbf{b}) = \left[\begin{array}{cc} \frac{x^2-y^2}{(x^2+y^2)^2} & \frac{2xy}{(x^2+y^2)^2} \\ \frac{2xy}{(x^2+y^2)^2} & \frac{y^2-x^2}{(x^2+y^2)^2} \end{array} \right]_{|(\mathbf{a},\mathbf{b})} = \left[\begin{array}{cc} 0 & 1/2 \\ 1/2 & 0 \end{array} \right] = -\frac{1}{4}$$

Since the determinant of this matrix is non-zero, the Implicit Function Theorem guarantees the existence of a neighborhood W around $(1/2,1/2)$ and a U containing $(\mathbf{a},\mathbf{b})$ such that each $(u,v) \in W$ corresponds to a unique (x,y) with $((x,y),(u,v)) \in U$ and $f((x,y),(u,v)) = 0$.

(d) Compute the Jacobian of the implicit function.

Solution

$$\left(\begin{array}{c} \frac{\partial g}{\partial u} \\ \frac{\partial g}{\partial v} \end{array} \right) = -\left[\begin{array}{cc} 0 & 1/2 \\ 1/2 & 0 \end{array} \right]^{-1} \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] = \left[\begin{array}{cc} 0 & -2 \\ -2 & 0 \end{array} \right]$$

3. Prove that there exist functions $u, v : \mathbb{R}^4 \longrightarrow \mathbb{R}$, continuously differentiable on some open neighborhood around the point $(x,y,z,w) = (2,1,-1,2)$

such that $u(2,1,-1,2)=4$ and $v(2,1,-1,2)=3$ and the equations

$$u^2+v^2+w^2=29 \text{ and } \frac{u^2}{x^2}+\frac{v^2}{y^2}+\frac{w^2}{z^2}=17$$

both hold for all (x,y,z,w) in that neighborhood.

Solution First, we need to check that our two equations hold at $(x,y,z,w,u,v)=(2,1,-1,2,4,3)$. Plugging the appropriate values into each equation, we see that $4^2+3^2+2^2=29$, and $\frac{4^2}{2^2}+\frac{3^2}{1^2}+\frac{2^2}{(-1)^2}=17$. So the equations hold at $(x,y,z,w)=(2,1,-1,2)$, a point which we will henceforth call s^* We shall also label $u^*=4$ and $v^*=3$.

To determine whether our functions u,v exist on some neighborhood around s^*, we must set up our system of equations in a way that is useful for the implicit function theroem. Define $F:\mathbb{R}^6 \longrightarrow \mathbb{R}^2$ so that:

$$F(x,y,z,w,u,v)=\begin{pmatrix} u^2+v^2+w^2-29 \\ \frac{u^2}{x^2}+\frac{v^2}{y^2}+\frac{w^2}{z^2}-17 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This is something to which we can apply the implicit function theorem. Note that the dimension of the range of F, which is 2, is equal to the number of endogenous variables (also 2, u and v). So, to verify whether or not we can express the endogenous variables as functions of the exogenous variables at s^*, we must verify that the determinant of the Jacobian derivative matrix of F with respect to the endogenous variables u and v is non-singular. That is, we must verify:

$$\begin{aligned}
|D_{u,v}F(s^*,u^*,v^*)| &\neq 0 \\
\Leftrightarrow \left|\begin{bmatrix} \frac{\partial f_1(s^*,u^*,v^*)}{\partial u} & \frac{\partial f_1(s^*,u^*,v^*)}{\partial v} \\ \frac{\partial f_2(s^*,u^*,v^*)}{\partial u} & \frac{\partial f_2(s^*,u^*,v^*)}{\partial v} \end{bmatrix}\right| &\neq 0 \\
\Leftrightarrow \left|\begin{bmatrix} 2u^* & 2v^* \\ \frac{2u^*}{(x^*)^2} & \frac{2v^*}{(y^*)^2} \end{bmatrix}\right| &\neq 0 \\
\Leftrightarrow \left|\begin{bmatrix} 8 & 6 \\ 2 & 6 \end{bmatrix}\right| &\neq 0 \\
\Leftrightarrow 36 &\neq 0
\end{aligned}$$

So the Jacobian matrix is non-singular, and we can define our fuvnctions u and v on a neighborhood of s^*.

4. Let $E=\{(x,y):0<y<x\}$ and set $f(x,y)=(x+y,xy)$ for $(x,y)\in E$.

   (a) Prove f is one-to-one from E onto $\{(s,t):s>2\sqrt{t},\ t>0\}$ and find a formula for $f^{-1}(s,t)$.

   Solution First, to prove that f is one-to-one from its entire domain of E, we must verify that the conditions of the inverse function theorem hold for all of E; that is, the determinant of the Jacobian matrix of partial derivatives of f must be non-singular at all points in E. We have:

$$\begin{aligned} |Df(x,y)| = \begin{vmatrix} \frac{\partial f_1(x,y)}{\partial x} & \frac{\partial f_1(x,y)}{\partial y} \\ \frac{\partial f_2(x,y)}{\partial x} & \frac{\partial f_2(x,y)}{\partial y} \end{vmatrix} &= \begin{vmatrix} 1 & 1 \\ y & x \end{vmatrix} \\ &= x - y \\ &\neq 0 \ \forall x, y \in E \text{ since } x > y \end{aligned}$$

So f is one-to-one from E onto $\{(s,t) : s > 2\sqrt{t},\ t > 0\}$. To find the formula of the inverse function, we must do some algebra to express x and y as functions of s and t. Letting $y = s - x$, we have that $t = xy = x(s - x)$. Solving this for x using the quadratic formula yields:

$$x = \frac{s \pm \sqrt{s^2 - 4t}}{2}$$

Which will always be real and positive given that $s > 2\sqrt{t}$, and $t > 0$. Using $y = s - x$ to solve for y yields:

$$y = \frac{s \mp \sqrt{s^2 - 4t}}{2}$$

This initally appears to violate the just-proven fact that f is one-to-one. However, noting that we must have $0 < y < x$, it must be that the formula for x takes the positve sign, while the formula for y takes the negative sign. That is:

$$x = \frac{s + \sqrt{s^2 - 4t}}{2},\ y = \frac{s - \sqrt{s^2 - 4t}}{2}$$

Which defines $f^{-1}(s,t)$.

(b) Use the inverse function theorem to compute $D(f^{-1})(f(x,y))$ for $x \neq y$.

Solution The IFT tells us that $D(f^{-1})(f(x,y)) = (Df(x,y))^{-1}$. Therefore:

$$\begin{aligned} D(f^{-1})(f(x,y)) &= \begin{bmatrix} 1 & 1 \\ y & x \end{bmatrix}^{-1} \\ &= \frac{1}{x-y} \begin{bmatrix} x & -1 \\ -y & 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{x}{x-y} & \frac{-1}{x-y} \\ \frac{-y}{x-y} & \frac{1}{x-y} \end{bmatrix} \end{aligned}$$

(c) Compare the two expressions for $D(f^{-1})(f(x,y))$ that you derived directly of using the Implicit Function Theorem

Solution To compute $D(f^{-1})(f(x,y))$ directly, we use our formula from part (i): $x = \frac{s+\sqrt{s^2-4t}}{2}$, $y = \frac{s-\sqrt{s^2-4t}}{2}$. Taking derivatives, we have that:

$$
\begin{aligned}
D(f^{-1})(f(x,y)) = D(f^{-1})(s,t) &= \frac{1}{2}\begin{bmatrix} 1+\frac{s}{\sqrt{s^2-4t}} & \frac{-2}{\sqrt{s^2-4t}} \\ 1-\frac{s}{\sqrt{s^2-4t}} & \frac{2}{\sqrt{s^2-4t}} \end{bmatrix} \\
&= \frac{1}{2}\frac{1}{\sqrt{s^2-4t}}\begin{bmatrix} \sqrt{s^2-4t}+s & -2 \\ \sqrt{s^2-4t}-s & 2 \end{bmatrix}
\end{aligned}
$$

To put this into terms of x and y, we use our inital $s = x+y$ and $t = xy$. This gives us that $\sqrt{s^2-4t} = \sqrt{(x+y)^2-4xy} = \sqrt{x^2+2xy+y^2-4xy} = \sqrt{(x-y)^2} = x-y$. Substituting everything into the above yields:

$$
\begin{aligned}
D(f^{-1})(f(x,y)) &= \frac{1}{2(x-y)}\begin{bmatrix} x-y+x+y & -2 \\ x-y-x-y & 2 \end{bmatrix} \\
&= \frac{1}{2(x-y)}\begin{bmatrix} 2x & -2 \\ -2y & 2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{x}{x-y} & \frac{-1}{x-y} \\ \frac{-y}{x-y} & \frac{1}{x-y} \end{bmatrix}
\end{aligned}
$$

Which is exactly what we had in part (ii). Once again, isn't it nice when math works out?

5. Consider the following system of first order differential equations:

$$
\begin{aligned}
\dot{x} &= x^{1/4} - y \\
\dot{y} &= y[\tfrac{3}{2}x^{-2/3} - \tfrac{1}{10}]
\end{aligned}
$$

(a) Plot the $\dot{x} = 0$ and $\dot{y} = 0$ loci for $x > 0$ in a phase diagram. Show the steady state, the direction of motion, and the approximate location of the stable and unstable arms.

Solution The $\dot{x} = 0$ locus is $y = x^{1/4}$ and the $\dot{y} = 0$ locus is both the horizontal line $y = 0$ and the vertical line $x = (1/15)^{-\frac{3}{2}}$. The steady state is at the intersection of the two loci, which occurs at $\left((1/15)^{(-3/2)}, (1/15)^{(-3/8)}\right)$ or approximately $(58.1, 2.76)$. Solutions move to the left above the $\dot{x} = 0$ locus and the right below. Solutions move upward on the left of the $\dot{y} = 0$ locus and downward on the right. Thus, paths look like hyperbolas.

(b) Linearize the system using a Taylor-series expansion around the $x > 0$ steady state. Write down the linearized equations.

Solution We will call the steady state (x^*, y^*) and calculate the Taylor expansion around that point. To make life simpler, we will define a new variable, $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} x - x^* \\ y - y^* \end{pmatrix}$, which translates the steady state to the origin. First, from the Taylor formula we have

$$\begin{aligned} \dot{x} &= \tfrac{1}{4}(x^*)^{-3/4}(x - x^*) - (y - y^*) \\ \dot{y} &= \left[\tfrac{3}{2}(-\tfrac{2}{3})(x^*)^{-5/3}\right](x - x^*) - 0 \cdot (y - y^*). \end{aligned}$$

Notice that since we are expanding around the steady state, the first term drops out and all we are left with is the first derivative term. On the second line, the zero comes from the fact that at the steady state, the expression in the brackets in the non-linear equatio n for $\dot{y}$ is zero. Now let's plug in the rewrite this system in terms of z and then plug in the values for (x^*, y^*).

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} 0.012z_1 - z_2 \\ -0.003z_1 \end{pmatrix} = \begin{pmatrix} 0.012 & -1 \\ -0.003 & 0 \end{pmatrix} z.$$

(c) Plot a phase diagram for the linearized system and compare the behavior at the steady state of the two systems.

Solution The $\dot{z}_1 = 0$ locus is $z_2 = 0.012z_1$ and the $\dot{z}_2 = 0$ locus is the line $z_1 = 0$. This gives us a steady state at the origin, which makes sense, because we have translated the steady state to the origin. Above the $\dot{z}_1 = 0$ locus, solutions move to the left and below it they move to the right. To the left of the $\dot{z}_2 = 0$ locus, solutions move upward and they move downward to the right. The slope the $\dot{z}_1 = 0$ locus should correspond to the slope of the $\dot{x} = 0$ locus at the steady state in the non-linear system. Both the $\dot{z}_2 = 0$ and $\dot{y} = 0$ loci are vertical lines. Checking the eigenvalues of the matrix of coefficients, we get two real eigenvalues of opposite sign:

$$\lambda_{1,2} = \frac{0.012 \pm \sqrt{0.012^2 - 4(-0.003)}}{2} = (0.06, -0.05).$$

We conclude from this that in the region around the steady state, the paths of the solutions look like hyperbolas. Thus, the behavior of the linearized system is qualitatively the same as the nonlinear system.

(d) Give the general solution of the linearized system.

Solution Two eigenvectors corresponding to λ_1 and λ_2 are $v_1 = \begin{pmatrix} 1 \\ -0.048 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ 0.062 \end{pmatrix}$, respectively. The general solutions then is $z(t) = C_1 e^{0.06t} v_1 + C_2 e^{-0.05t} v_2$, or changing back to

(x, y) coordinates,

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{0.06t} \begin{pmatrix} 1 \\ -0.048 \end{pmatrix} + C_2 e^{-0.05t} \begin{pmatrix} 1 \\ 0.062 \end{pmatrix} + \begin{pmatrix} x^* \\ y^* \end{pmatrix}$$

6. Consider the second order linear differential equation given by $y'' = -y - y'$.

(a) Show how this equation can be rewritten as the following *first* order linear differential equation of two variables:

$$\bar{x}'(t) = A\bar{x}(t),$$

where $A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$ and $\bar{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$.

Solution Define the new variable $\bar{x} = \begin{bmatrix} y \\ y' \end{bmatrix}$. This gives us

$$x' = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} y' \\ -y - y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} = Ax.$$

(b) Describe the solutions of the first order system (verbally) by analyzing the matrix A.

Solution The eigenvalues of A are $\lambda_{1,2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. Because the eigenvalues are complex, the solutions to the system spiral around the origin. Because the real parts of the solutions are both negative, the sytem spirals inward, converging to a steady state at the origin.

(c) In a phase diagram, show the behavior of the system using the previous analysis and by solving for $x_1'(t) = 0$ and $x_2'(t) = 0$.

Solution The $x_1' = 0$ locus is the line $x_2 = 0$, that is the horizontal axis. All paths of the system cross this line vertically. Above this line, the solution moves from left to right and below it solutions move from right to left. The $x_2' = 0$ locus is the line $x_2 = -x_1$ and all paths cross this line horizontally. Above this line solutions move from up to down and below the line they move from down to up. The intersection of these two line yields a stable steady state at the origin. In summary, the solutions spiral inwards around the origin in a clockwise direction.

(d) Give the solution of the system when $x_1(t_0) = 0$ and $x_2'(t_0) = 1$.

Solution Instead of carrying the expression $(t - t_0)$ through this solution, we will simply let $t_0 = 0$. The eigenvectors corresponding to λ_1 and λ_2 are $v_1 = u + iw$ and $v_2 = u - iw$, where $u = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ and $w = \begin{pmatrix} 0 \\ -\sqrt{3} \end{pmatrix}$. Expressed in terms of the eigenvector basis, the solution takes the following form:

$$\begin{array}{lll} z_1 & = C_1 e^{\lambda_1 t} = C_1 e^{-\frac{t}{2}} e^{i\frac{\sqrt{3}}{2}} = & C_1 e^{-\frac{t}{2}}(\cos(\frac{t\sqrt{3}}{2}) + i\sin(\frac{t\sqrt{3}}{2})) \\ z_2 & = C_2 e^{\lambda_2 t} = C_2 e^{-\frac{t}{2}} e^{-i\frac{\sqrt{3}}{2}} = & C_2 e^{-\frac{t}{2}}(\cos(\frac{t\sqrt{3}}{2}) - i\sin(\frac{t\sqrt{3}}{2})) \end{array}.$$

We can rewrite this solution in terms of the standard basis and strictly real coordinates by pre-multiplying by P, the matrix that has the standard coordinates of v_1 and v_2 as columns. This gives us

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = e^{-\frac{t}{2}} \left[(d_1 u - d_2 w)\cos(\frac{t\sqrt{3}}{2}) - (d_2 u + d_1 w)\sin(\frac{t\sqrt{3}}{2}) \right],$$

where d_1, d_2 are constants. Check you section notes to see how these constants relate to C_1 and C_2. Plugging in our values of u and w from above, we get

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} e^{-\frac{t}{2}} \left[-2d_1 \cos(\frac{t\sqrt{3}}{2}) + d_2 \sin(\frac{t\sqrt{3}}{2}) \right] \\ e^{-\frac{t}{2}} \left[(d_1 + d_2\sqrt{3})\cos(\frac{t\sqrt{3}}{2}) - (d_2 - d_1\sqrt{3})\sin(\frac{t\sqrt{3}}{2}) \right] \end{pmatrix}.$$

We can solve for d_1 and d_2 using our boundary conditions, without bothering with the C constants. We begin by evaluating $x_1(0)$, which makes the sine and cosine expressions disappear and leaves us with $x_1(0) = d_1(-2) = 0$ or $d_1 = 0$. Since $d_1 = 0$, we can rewrite $x_2(t)$ more simply:

$$x_2(t) = e^{-\frac{t}{2}} \left[d_2\sqrt{3}\cos(\frac{t\sqrt{3}}{2}) - (d_2 \sin(\frac{t\sqrt{3}}{2}) \right].$$

Our second boundary condition is that $x'(0) = 1$, so we (carefully) take the derivative, evaluate at zero, and set equal to one. This yields $d_2 = -1/\sqrt{3}$. Substituting in these value of d_1 and d_2, we write the solution as

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \frac{e^{-\frac{t}{2}}}{\sqrt{3}} \left[u\sin(\frac{t\sqrt{3}}{2}) - w\cos(\frac{t\sqrt{3}}{2}) \right].$$